

Propagation of signals in spaces with affine connections and metrics

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Abstract

The propagation of signals in space-time is considered on the basis of the notion of null (isotropic) vector field in spaces with affine connections and metrics $[(\bar{L}_n, g)$ -spaces] as models of space or space-time. The Doppler effect is generalized for these types of spaces. The notions of aberration, standard (longitudinal) Doppler effect, and transversal Doppler effect are introduced. On their grounds, the Hubble effect appears as Doppler effect with explicit forms of the centrifugal (centripetal) and Coriolis velocities and accelerations in spaces with affine connections and metrics. Doppler effect, Hubble effect, and aberration could be used in mechanics of continuous media and in other classical field theories in the same way as the standard Doppler effect is used in classical and relativistic mechanics.

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1 Introduction

1. Modern problems of relativistic astrophysics as well as of relativistic physics (dark matter, dark energy, evolution of the universe, measurement of velocities of moving objects etc.) are related to the propagation of signals in space or in space-time. The basis of experimental data received as results of observations of the Doppler effect or of the Hubble effect gives rise to theoretical considerations about the theoretical status of effects related to detecting signals from emitters moving relatively to observers carrying detectors in their laboratories.

2. In classical physics, the Doppler effect is considered only as a longitudinal effect in a continuous media or in vacuum. It is assumed that the reason for this effect is the relative velocity between emitter and observer. In acoustics, the signals are considered as propagating in a continuous media.

3. In relativistic physics, the Doppler effect is considered in special relativity as (standard) longitudinal and transversal effect caused by the motion of electromagnetic signals in vacuum with respect to an observer (detector). In general relativity, the Doppler effect is related to the propagation of light in

astrophysical systems and to the existence of the red shift relation due to the Hubble effect and Hubble law. The question arises if the theoretical basis for the use of Doppler effect and Hubble effect as tools for check-up of theoretical models in astrophysics and relativistic physics in sophisticated models of space-time such as spaces with affine connections and metrics $[(\overline{L}_n, g)$ -spaces] is sufficiently work out. It has been recently shown that every classical (non-quantized) field theory could be considered as a theory of continuous media [1] ÷ [4]. On this basis, the propagation of signals in different models of space or of space-time is worth being investigated. From this point of view, questions arise as how Doppler effect is related to the Hubble effect from point of view of the kinematic characteristics of a continuous media and, especially, is there a relation between the Doppler and Hubble effects and the relative accelerations between emitters and detectors. In a previous paper [5] Doppler and Hubble effects are considered on the basis of dimension preconditions with relations to the relative velocity between an emitter and an observer. In this paper we will use the properties of a covariant exponential operator for finding out the change of a null (isotropic) vector field along the world line of an observer (detector). It is assumed that the null vector field is related to the propagation of a signal when at a given time moment in the frame of the observer the emitter and the observer are at rest. After that moment the observer could detect the relative motion of the emitter and observe the frequency shifts of its signals.

4. The notion of null (isotropic) vector field is related to the light propagation described in relativistic electrodynamics on the basis of special and general relativity theories [6] ÷ [8]. On the other side, the notion of null (isotropic) vector field could be considered in spaces with (definite) or indefinite metric as a geometric object (contravariant vector field) with specific properties making it useful in the description of the propagation of signals in space or in space-time as well as in geometrical optics based on different mathematical models. Usually, it is assumed that a signal is propagating with limited velocity through a continuous media or in vacuum. The velocity of propagation of signals could be a constant quantity or a non-constant quantity depending on the properties of the space or the space-time, where the signals are transmitted and propagated.

5. In the present paper the notion of contravariant null (isotropic) vector field is introduced and considered in spaces with affine connections and metrics $[(\overline{L}_n, g)$ -spaces]. In Section 2 the properties of null vector fields are considered on the basis of $(n - 1) + 1$ representation of non-null (non-isotropic) vector fields orthogonal to each other. In Section 3 the notions of distance and space velocity are discussed and their relations to null vector fields are investigated. In Section 4 the kinematic effects [aberration, longitudinal and transversal Doppler effects, and Hubble effect] related to the kinematic characteristics of the relative velocity and the relative acceleration as well as their connections with null vector fields are considered. In Section 4 the kinematic effects related to the relative velocity and the relative acceleration are recalled. In Section 5 the aberration of signals is considered as corollary of the change of a null vector field along the world line of an observer. In Section 6 the different types of Doppler effect are introduced and investigated.. In Section 7 the Hubble effect as Doppler effect with explicitly given forms of the relative velocities and the relative accelerations is considered. It is shown that the Hubble effect appears as a corollary of the standard (longitudinal) and transversal Doppler effects. On the other side, the Hubble effect is closely related to centrifugal (centripetal) and Coriolis velocities

and accelerations. The results discussed in the paper could be important from the point of view of the possible applications of kinematic characteristics in continuous media mechanics as well as in classical (non-quantum) field theories in spaces with affine connections and metrics. Section 8 comprises concluding remarks.

The main results in the paper are given in details (even in full details) for these readers who are not familiar with the considered problems. The definitions and abbreviations are identical to those used in [3] and [4]. The reader is kindly asked to refer to them for more details and explanations of the statements and results only cited in this paper.

2 Null (isotropic) vector fields. Definition and properties

2.1 Definition of a null (isotropic) vector field

Let us now consider a space with affine connections and metrics $[(\overline{L}_n, g)\text{-space}]$ [9], [10] as a model of a space or of a space-time. In this space the length l_v of a contravariant vector field $v \in T(M)$ is defined by the use of the covariant metric tensor field (covariant metric) $g \in \otimes_{2s}(M)$ as

$$g(v, v) = \pm l_v^2 \quad , \quad l_v^2 \geq 0 \quad . \quad (1)$$

Remark. The sign before l_v^2 depends on the signature Sgn of the covariant metric g . M is differentiable manifold, $\dim M = n$. A (\overline{L}_n, g) -space is a differentiable manifold M provided with contravariant and covariant affine connections (whose components differ not only by sign) and metrics. $T(M) = \cup_{x \in M} T_x(M)$. $T_x(M)$ is the tangent space at a point $x \in M$. $\otimes_{2s}(M)$ is the space of covariant symmetric tensors g of second rank with $\det g \neq 0$ over M .

The contravariant vector fields can be divided into two classes with respect to their lengths:

- null or isotropic vector fields with length $l_v = 0$,
- non-null or non-isotropic vector fields with length $l_v \neq 0$.

In the case of a positive definite covariant metric g ($Sgng = \pm n$, $\dim M = n$) the null (isotropic) vector field is identically equal to zero, i.e. if $l_v = 0$ then $v = v^i \cdot e_i \equiv \mathbf{0} \in T(M)$, $v^i \equiv 0$.

In the case of an indefinite covariant metric g ($Sgng < n$ or $Sgng > -n$, $\dim M = n$) the null (isotropic) vector field with equal to zero length $l_v = 0$ can have different from zero components in an arbitrary given basis, i.e. it is not identically equal to zero at the points, where it has been defined, i.e. if $l_v = 0$ then $v \neq \mathbf{0} \in T(M)$, $v = v^i \cdot e_i \in T(M)$ and $v^i \neq 0$. In a (\overline{L}_n, g) -space the components g_{ij} of a covariant metric tensor g could be written in a local co-ordinate system at a given point of the space as $g_{ij} = \underbrace{(-1, -1, -1, \dots, +1, +1, +1, \dots)}_{k \text{ times} \quad l \text{ times}}$

with $k + l = n$.

The signature Sgn of g is defined as

$$Sgn g = -k + l = 2 \cdot l - n = n - 2 \cdot k \quad , \quad n, k, l \in \mathbf{N}, \quad (2)$$

where $k = n - l$, $l = n - k$.

In the relativistic physics for $\dim M = 4$, the number l and k are chosen as $l = 1$, $k = 3$ or $l = 3$, $k = 1$ so that $Sgn g = -2 \sim (-1, -1, -1, +1)$ or $Sgn g = +2 \sim (+1, +1, +1, -1)$. In general, a (\bar{L}_n, g) -space could be consider as a model of space-time with $Sgn g < 0$ and $(k > l, l = 1)$ or with $Sgn g > 0$ and $(l > k, k = 1)$.

The non-null (non-isotropic) contravariant vector fields are divided into two classes.

1. For $Sgn g < 0$

(a) $g(v, v) = +l_v^2 > 0$:= time like vector field $v \in T(M)$,

(b) $g(v, v) = -l_v^2 < 0$:= space like vector field $v \in T(M)$.

2. For $Sgn g > 0$

(a) $g(v, v) = -l_v^2 < 0$:= time like vector field $v \in T(M)$,

(b) $g(v, v) = +l_v^2 > 0$:= space like vector field $v \in T(M)$.

Therefore, if we do not fix a priory the signature of the space-time models we can distinguish a *time like vector field* u with

$$\begin{aligned} g(u, u) &= +l_u^2 & \text{for } Sgn g < 0 \\ &= -l_u^2 & \text{for } Sgn g > 0 \end{aligned}$$

or $g(u, u) = \pm l_u^2$, and a *space like vector field* ξ_\perp with

$$\begin{aligned} g(\xi_\perp, \xi_\perp) &= -l_{\xi_\perp}^2 & \text{for } Sgn g < 0 \\ &= +l_{\xi_\perp}^2 & \text{for } Sgn g > 0 \end{aligned}$$

or $g(\xi_\perp, \xi_\perp) = \mp l_{\xi_\perp}^2$. This means that in symbols $\pm l_\diamond^2$ or $\mp l_\diamond^2$ ($\diamond \in T(M)$) the sign above is related to $Sgn g < 0$ and the sign below is related to $Sgn g > 0$.

Remark. Since $l_\diamond = \pm \sqrt{l_\diamond^2}$, the sings in this case will be denoted as not related to the signature of the metric g . Usually, it is assumed that $l_\diamond = +\sqrt{l_\diamond^2} \geq 0$.

A non-null (non-isotropic) contravariant vector field v could be represented by its length l_v and its corresponding unit vector $n_v = \frac{v}{l_v}$ as $v = \pm l_v \cdot n_v$ in contrast to a null (isotropic) vector field \tilde{k} with $l_{\tilde{k}} = 0$ (the sings here are not related to the signature of the metric g)

$$v = \pm l_v \cdot n_v \quad , \quad g(v, v) = l_v^2 \cdot g(n_v, n_v) = \pm l_v^2 \quad , \quad g(n_v, n_v) = \pm 1 \quad ,$$

for time like unit vector field n_v or

$$v = \mp l_v \cdot n_v \quad , \quad g(v, v) = l_v^2 \cdot g(n_v, n_v) = \mp l_v^2 \quad , \quad g(n_v, n_v) = \mp 1 \quad ,$$

for space like unit vector field n_v .

Remark. In the experimental physics, the measurements are related to the lengths and to the directions of a non-null (non-isotropic) vector field with respect to a frame of reference. Since different types of co-ordinates could be used in a frame of reference, the components of a vector field related to these co-ordinates cannot be considered as invariant characteristics of the vector field and on this grounds the components cannot be important characteristics for the vector fields.

After these preliminary remarks, we can introduce the notion of a null (isotropic) vector field

Definition 1. A contravariant vector field \tilde{k} with length zero is called null (isotropic) vector field, i.e. $\tilde{k} := \text{null (isotropic) vector field}$ if

$$g(\tilde{k}, \tilde{k}) = \pm l_k^2 = 0 \quad , \quad l_{\tilde{k}} = |g(\tilde{k}, \tilde{k})|^{1/2} = 0 \quad . \quad (3)$$

2.2 Properties of a null (isotropic) vector field

The properties of a null (isotropic) contravariant vector field could be considered in a $(n-1)+1$ invariant decomposition of a space-time by the use of two non-isotropic contravariant vector fields u and ξ_\perp , orthogonal to each other [9], i.e. $g(u, \xi_\perp) = 0$. The contravariant vectors u and ξ_\perp are essential elements of the structure of a frame of reference [11] in a space-time.

2.2.1 Invariant representation of a null vector field by the use of a non-null contravariant vector field

(a) *Invariant projections of a null vector field along and orthogonal to a non-null (non-isotropic) contravariant vector field u*

Every contravariant vector field $\tilde{k} \in T(M)$ could be represented in the form

$$\tilde{k} = \frac{1}{e} \cdot g(u, \tilde{k}) \cdot u + \bar{g}[h_u(\tilde{k})] = k_\parallel + k_\perp \quad , \quad (4)$$

where

$$\begin{aligned} e &= g(u, u) = \pm l_u^2 \quad , \\ \bar{g} &= g^{ij} \cdot \partial_i \cdot \partial_j \quad , \quad \partial_i \cdot \partial_j = \frac{1}{2} \cdot (\partial_i \otimes \partial_j + \partial_j \otimes \partial_i) \quad , \\ g &= g_{ij} \cdot dx^i \cdot dx^j \quad , \quad dx^i \cdot dx^j = \frac{1}{2} \cdot (dx^i \otimes dx^j + dx^j \otimes dx^i) \quad , \\ h_u &= g - \frac{1}{e} \cdot g(u) \otimes g(u) \quad , \quad h^u = \bar{g} - \frac{1}{e} \cdot u \otimes u \quad , \\ g(u) &= g_{i\bar{j}} \cdot u^{\bar{j}} \cdot dx^i \quad , \\ \bar{g}[h_u(\tilde{k})] &= g^{ij} \cdot h_{j\bar{l}} \cdot \tilde{k}^{\bar{l}} \cdot \partial_i := k_\perp \quad , \quad k_\parallel := \frac{1}{e} \cdot g(u, \tilde{k}) \cdot u \quad . \end{aligned} \quad (5)$$

$$g(k_\parallel, k_\perp) = 0 \quad , \quad g(u, k_\perp) = 0 \quad . \quad (6)$$

Let us now take a closer look at the first term k_\parallel of the representation of \tilde{k} .

$$k_\parallel = \frac{1}{e} \cdot g(u, \tilde{k}) \cdot u = \pm \frac{1}{l_u^2} \cdot g(u, \tilde{k}) \cdot u = \pm \frac{1}{l_u} \cdot g(u, k_\parallel) \cdot \frac{u}{l_u} \quad . \quad (7)$$

If we introduce the abbreviations

$$n_\parallel = \frac{u}{l_u} \quad , \quad \omega = g(u, \tilde{k}) = g(u, k_\parallel) \quad , \quad (8)$$

where

$$g(n_\parallel, n_\parallel) = \frac{1}{l_u^2} \cdot g(u, u) = \frac{1}{l_u^2} \cdot (\pm l_u^2) = \pm 1 \quad , \quad (9)$$

$$\begin{aligned}\omega &= g(u, \tilde{k}) = g(u, k_{\parallel} + k_{\perp}) = g(u, k_{\parallel}) = l_u \cdot g(n_{\parallel}, k_{\parallel}) = \\ &= l_u \cdot g(k_{\parallel}, n_{\parallel}) ,\end{aligned}\quad (10)$$

$$\begin{aligned}k_{\parallel} &: = \pm l_{k_{\parallel}} \cdot n_{\parallel} \quad , \quad g(k_{\parallel}, k_{\parallel}) = l_{k_{\parallel}}^2 \cdot g(n_{\parallel}, n_{\parallel}) = \\ &= l_{k_{\parallel}}^2 \cdot (\pm 1) = \pm l_{k_{\parallel}}^2 \quad ,\end{aligned}\quad (11)$$

$$g(k_{\parallel}, n_{\parallel}) = \pm l_{k_{\parallel}} \cdot g(n_{\parallel}, n_{\parallel}) = \pm l_{k_{\parallel}} \cdot (\pm 1) = l_{k_{\parallel}} = \frac{\omega}{l_u} \quad ,\quad (12)$$

then k_{\parallel} could be expressed as (the signs are not related to the signature of the metric g)

$$k_{\parallel} = \pm \frac{\omega}{l_u} \cdot n_{\parallel} = \pm l_{k_{\parallel}} \cdot n_{\parallel} \quad .\quad (13)$$

The vector n_{\parallel} is a unit vector [$g(n_{\parallel}, n_{\parallel}) = \pm 1$] collinear to u and, therefore, tangential to a curve with parameter τ if $u = \frac{d}{d\tau}$.

The scalar invariant $\omega = g(u, \tilde{k})$ is usually interpreted as the frequency of the radiation related to the null vector field \tilde{k} and propagating with velocity u with absolute value l_u with respect to the trajectory $x^i(\tau)$. In general relativity $l_u := c$ and it is assumed that the radiation is of electromagnetic nature propagating with the velocity of light c in vacuum. We will come back to this interpretation in the next considerations.

The contravariant vector field k_{\perp}

$$k_{\perp} = \bar{g}[h_u(\tilde{k})]$$

is orthogonal to u (and k_{\parallel} respectively) part of \tilde{k} . Since

$$g(k_{\parallel}, k_{\parallel}) = g(\pm \frac{\omega}{l_u} \cdot n_{\parallel}, \pm \frac{\omega}{l_u} \cdot n_{\parallel}) = \frac{\omega^2}{l_u^2} \cdot g(n_{\parallel}, n_{\parallel}) = \pm \frac{\omega^2}{l_u^2} = \pm l_{k_{\parallel}}^2 \quad ,\quad (14)$$

$$l_{k_{\parallel}} = \frac{\omega}{l_u} > 0 \quad , \quad l_{k_{\parallel}}^2 = \frac{\omega^2}{l_u^2} \quad ,\quad (15)$$

and

$$g(\tilde{k}, \tilde{k}) = 0 \quad , \quad g(k_{\parallel}, k_{\perp}) = 0 \quad ,$$

we have for $g(k_{\perp}, k_{\perp})$

$$\begin{aligned}g(\tilde{k}, \tilde{k}) &= 0 = g(k_{\parallel} + k_{\perp}, k_{\parallel} + k_{\perp}) = g(k_{\parallel}, k_{\parallel}) + g(k_{\perp}, k_{\perp}) = \\ &= \pm \frac{\omega^2}{l_u^2} + g(k_{\perp}, k_{\perp}) = \pm \frac{\omega^2}{l_u^2} \mp l_{k_{\perp}}^2 = 0 \quad ,\end{aligned}\quad (16)$$

and, therefore,

$$l_{k_{\perp}}^2 = \frac{\omega^2}{l_u^2} \quad , \quad l_{k_{\perp}} = \frac{\omega}{l_u} = l_{k_{\parallel}} \quad .\quad (17)$$

Remark. Since $\omega \geq 0$ and $l_u > 0$, and at the same time $l_{k_{\perp}} > 0$, and $l_{k_{\parallel}} > 0$, we have

$$l_{k_{\parallel}} = \frac{\omega}{l_u} = l_{k_{\perp}} \quad .$$

If we introduce the unit contravariant vector \tilde{n}_{\perp} with $g(\tilde{n}_{\perp}, \tilde{n}_{\perp}) = \mp 1$ then the vector k_{\perp} could be written as

$$k_{\perp} := \mp l_{k_{\perp}} \cdot \tilde{n}_{\perp} \quad ,\quad (18)$$

where

$$g(k_\perp, k_\perp) = l_{k_\perp}^2 \cdot g(\tilde{n}_\perp, \tilde{n}) = \mp l_{k_\perp}^2 = \mp \frac{\omega^2}{l_u^2} \quad , \quad l_{k_\perp}^2 = \frac{\omega^2}{l_u^2} \quad , \quad l_{k_\perp} = \frac{\omega}{l_u} \quad . \quad (19)$$

Therefore,

$$k_\perp = \mp \frac{\omega}{l_u} \cdot \tilde{n}_\perp \quad , \quad k_\parallel = \pm \frac{\omega}{l_u} \cdot n_\parallel \quad , \quad (20)$$

$$\tilde{k} = k_\parallel + k_\perp = \pm \frac{\omega}{l_u} \cdot (n_\parallel - \tilde{n}_\perp) \quad , \quad (21)$$

where

$$g(n_\parallel, \tilde{n}_\perp) = 0 \quad , \quad g(k_\parallel, \xi_\perp) = 0 \quad , \quad g(u, k_\perp) = 0 \quad , \quad (22)$$

$$g(n_\parallel, k_\parallel) = \pm l_{k_\parallel} \cdot g(n_\parallel, n_\parallel) = l_{k_\parallel} = \frac{\omega}{l_u} \quad , \quad (23)$$

$$g(\tilde{n}_\perp, k_\perp) = \mp l_{k_\perp} \cdot g(n_\perp, n_\perp) = \frac{\omega}{l_u} = l_{k_\parallel} = l_{k_\perp} \quad , \quad (24)$$

$$g(n_\parallel, k_\parallel) = g(\tilde{n}_\perp, k_\perp) = \frac{\omega}{l_u} = l_{k_\parallel} = l_{k_\perp} \quad . \quad (25)$$

Remark. The signs, not related to the metric g , are chosen so to be the same with the signs related to the metric g .

We have now the relations

$$\omega = g(u, \tilde{k}) = l_u \cdot g(n_\parallel, k_\parallel) = l_u \cdot g(\tilde{n}_\perp, k_\perp) \quad . \quad (26)$$

If \tilde{n}_\perp is interpreted as the unit vector in the direction of the propagation of a signal in the subspace orthogonal to the contravariant vector field u and l_u is interpreted as the absolute value of the velocity of the radiated signal then $l_u \cdot \tilde{n}_\perp$ is the path along \tilde{n}_\perp propagated by the signal in a unit time interval. Then

$$\omega = g(u_\perp, k_\perp) \quad , \quad u_\perp := l_u \cdot \tilde{n}_\perp \quad , \quad g(u, u_\perp) = 0 \quad . \quad (27)$$

2.2.2 Explicit form of the vector field k_\perp and its interpretation

Let us now consider more closely the explicit form of k_\perp

$$k_\perp = \mp l_{k_\perp} \cdot \tilde{n}_\perp = \mp \frac{\omega}{l_u} \cdot \tilde{n}_\perp \quad . \quad (28)$$

(a) In 3-dimensional Euclidean space (as model of space-time of the Newtonian mechanics) the wave vector \vec{k} is defined as

$$\vec{k} = \frac{2\pi}{\lambda} \cdot \vec{n} \quad , \quad (29)$$

where \vec{n} is the unit 3-vector in the direction of propagation of a signal with absolute value of its velocity $l_u = \lambda \cdot \nu$. If we express λ by $\lambda = l_u / \nu$ and put the equivalent expression in this for \vec{k} we obtain the expression

$$\vec{k} = \frac{2\pi \cdot \nu}{l_u} \cdot \vec{n} = \frac{\omega}{l_u} \cdot \vec{n} \quad , \quad (30)$$

which (up to a sign depending on the signature of the metric g) is identical with the expression for k_\perp for $n = 3$ if $k_\perp = \vec{k}$, $\vec{n} = \vec{n}_\perp$, and $\omega = 2 \cdot \pi \cdot \nu$.

(b) In 4-dimensional (pseudo) Riemannian space (as a model of space-time of the Einstein theory of gravitation) l_u is interpreted as the absolute value of the velocity of light in vacuum (normalized by some authors to 1), i.e. $l_u = c$, 1. Then

$$k_\perp = \mp \frac{\omega}{c} \cdot \vec{n}_\perp = \mp \frac{2 \cdot \pi \cdot \nu}{\lambda \cdot \nu} \cdot \vec{n}_\perp = \mp \frac{2 \cdot \pi}{\lambda} \cdot \vec{n}_\perp \quad (31)$$

and we obtain the expression for the wave vector of light propagation in general relativity, where \vec{n}_\perp is the unit vector along the propagation of light in the corresponding 3-dimensional subspace of an observer with world line $x^i(\tau)$ if

$$u = \frac{d}{d\tau} = l_u \cdot n_\parallel \quad , \quad n_\parallel = \frac{1}{l_u} \cdot \frac{d}{d\tau} \quad . \quad (32)$$

l_u is the velocity of light measured by the observer.

(c) In the general case for k_\perp as

$$k_\perp = \mp \frac{\omega}{l_u} \cdot \vec{n}_\perp \quad , \quad (33)$$

ω could also be interpreted as the frequency of a signal propagating with velocity with absolute value l_u in a frame of reference of an observer with world line $x^i(\tau)$. The unit vector \vec{n}_\perp is the unit vector in the direction of the propagation of the signal in the subspace orthogonal to the vector u . The velocity of the observer is usually defined by the use of the parameter τ of the world line under the assumption that $ds = l_u \cdot d\tau$, where ds is the distance of the propagation of a signal for the proper time interval $d\tau$ of the observer

$$u = \frac{d}{d\tau} = \frac{d}{\frac{1}{l_u} \cdot ds} = l_u \cdot \frac{d}{ds} \quad . \quad (34)$$

Remark. Usually the velocity of a particle (observer) moving in space-time is determined by its velocity vector field $u = \frac{d}{d\tau}$, where τ is the proper time of the observer. The parameter τ is considered as a parameter of observer's world line $x^i(\tau)$. By the use of u and its corresponding projection metrics h_u and h^u a contravariant (non-null, non-isotropic) vector field ξ could be represented in two parts: one part is collinear to u and the other part is orthogonal to u

$$\xi = \frac{1}{e} \cdot g(\xi, u) \cdot u + \bar{g}[h_u(\xi)] = \xi_\parallel + \xi_\perp \quad , \quad (35)$$

where

$$\xi_\parallel = \frac{1}{e} \cdot g(\xi, u) \cdot u \quad , \quad \xi_\perp = \bar{g}[h_u(\xi)] \quad , \quad g(\xi_\parallel, \xi_\perp) = 0 \quad . \quad (36)$$

1. If an observer is moving with velocity $v = \frac{d}{d\bar{\tau}}$ on his world line $x^i(\bar{\tau})$ then his velocity, considered with respect to the observer with velocity u and world line $x^i(\tau)$, will have two parts v_\parallel and v_\perp , collinear and orthogonal to u respectively at the cross point $\tau = \bar{\tau}$ of both the world lines $x^i(\tau)$ and $x^i(\bar{\tau})$

$$v = \frac{1}{e} \cdot g(v, u) \cdot u + \bar{g}[h_u(v)] = v_\parallel + v_\perp \quad . \quad (37)$$

The vector v_{\parallel} describes the motion of the observer with velocity v along the world line of the first observer with velocity u . The vector v_{\perp} describes the motion of the second observer with velocity v in direction orthogonal to the world line of the first observer. The vector v_{\perp} is the velocity of the second observer in the space of the first observer in contrast to the vector v_{\parallel} describing the change of v in the time of the first observer.

2. If we consider the propagation of a signal, characterized by its null vector field \tilde{k} , the interpretation of the vector field u , tangential to the world line of an observer, changes. The vector field $u = l_u \cdot n_{\parallel}$ is interpreted as the velocity vector field of the signal, propagating in the space-time and measured by the observer at its world line $x^i(\tau)$ with proper time τ as a parameter of this world line. The absolute value l_u of u is the size of the velocity of the signal measured along the unit vector field n_{\parallel} collinear to the tangent vector of the world line of the observer.

3. In Einstein's theory of gravitation (ETG) both interpretations of the vector field u are put together. On the one side, the vector field u is interpreted as the velocity of an observer on his world line with parameter τ interpreted as the proper time of the observer. On the other side, the length l_u of the vector field u is normalized either to ± 1 or to $\pm c = \text{const}$. The quantity c is interpreted as the light velocity in vacuum. The basic reason for this normalization is the possibility for normalization of every non-null (non-isotropic) vector field u in the form

$$n_u = \frac{u}{l_u} = n_{\parallel} \quad , \quad \text{where} \quad l_u = |g(u, u)|^{1/2} \neq 0 \quad , \quad (38)$$

by the use of its different from zero length ($l_u \neq 0$), defined by means of the covariant metric tensor g .

Both the interpretations of the vector field u (as a velocity of an observer or as velocity of a signal) should be considered separately from each other for avoiding ambiguities. The identification of the interpretations should mean that we assume the existence of an observer moving in space-time with velocity u and emitting (or receiving) signals with the same velocity. Such assumption does not exist in the Einstein theory of gravitation. This problem is worth to be investigated and a clear difference between both interpretations should be found. It is related to the notions of distance and of velocity in spaces with affine connections and metrics.

3 Distance and velocity in a (\bar{L}_n, g) -space

3.1 Distance in a (\bar{L}_n, g) -space and its relations to the notion of velocity

3.1.1 Distance in a (\bar{L}_n, g) -space for world and space lines not depending to each other

1. The distance in a (\bar{L}_n, g) -space between a point $P \in M$ with co-ordinates x^i and a point $\bar{P} \in M$ with co-ordinates $\bar{x}^i = x^i + dx^i$ is determined by the length of the ordinary differential d , considered as a contravariant vector field $d = dx^i \cdot \partial_i$ [3]. If we denote the distance as ds between point P and point \bar{P}

then the square ds^2 of ds could be defined as the square of the length of the ordinary differential d

$$ds^2 = g(d, d) = \pm l_d^2 = g_{\bar{i}\bar{j}} \cdot dx^i \cdot dx^j \quad , \quad l_d^2 \geq 0 \quad . \quad (39)$$

2. Let us now consider a two parametric congruence of curves (a set of not intersecting curves) in a (\bar{L}_n, g) -space

$$x^i = x^i(\tau, r(\tau, \lambda)) = x^i(\tau, \lambda) \quad , \quad (40)$$

where the function $r = r(\tau, \lambda) \in C^r(M)$, $r \geq 2$, depends on the two parameters τ and λ , $\tau, \lambda \in \mathbf{R}$. Then

$$dr = \frac{\partial r(\tau, \lambda)}{\partial \tau} \cdot d\tau + \frac{\partial r(\tau, \lambda)}{\partial \lambda} \cdot d\lambda \quad (41)$$

and

$$\begin{aligned} dx^i &= \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial \tau} \cdot d\tau + \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial r} \cdot \left(\frac{\partial r(\tau, \lambda)}{\partial \tau} \cdot d\tau + \right. \\ &\quad \left. + \frac{\partial r(\tau, \lambda)}{\partial \lambda} \cdot d\lambda \right) \\ &= \left[\frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial \tau} + \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial r} \cdot \frac{\partial r(\tau, \lambda)}{\partial \tau} \right] \cdot d\tau + \\ &\quad + \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial r} \cdot \frac{\partial r(\tau, \lambda)}{\partial \lambda} \cdot d\lambda \end{aligned} \quad (42)$$

or

$$dx^i = (u^i + \bar{\xi}^i \cdot l_v) \cdot d\tau + \bar{\xi}^i \cdot \frac{\partial r}{\partial \lambda} \cdot d\lambda \quad , \quad (43)$$

where

$$u^i = \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial \tau} \quad , \quad l_v = \frac{\partial r(\tau, \lambda)}{\partial \tau} \quad , \quad \bar{\xi}^i = \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial r} \quad , \quad (44)$$

and

$$\begin{aligned} d &= dx^i \cdot \partial_i = d\tau \cdot (u + l_v \cdot \bar{\xi}) + (\partial r / \partial \lambda) \cdot d\lambda \cdot \bar{\xi} \quad , \\ \frac{d\lambda}{d\tau} &= 0 \quad , \quad \frac{d\tau}{d\lambda} = 0 \quad . \end{aligned} \quad (45)$$

Remark. Here, the parameters τ and λ are assumed to be independent to each other functions.

The change of the contravariant vector field d under the change $d\tau$ of the parameter τ could be expressed in the form

$$\frac{d}{d\tau} = \frac{dx^i}{d\tau} \cdot \partial_i = u + l_v \cdot \bar{\xi} = \bar{u}^i \cdot \partial_i = \bar{u} \quad , \quad \bar{u}^i = \frac{dx^i}{d\tau} \quad , \quad (46)$$

where the relations are valid

$$\begin{aligned} g(\bar{u}, u) &= g(u, u) + l_v \cdot g(\bar{\xi}, u) \quad , \\ g(\bar{u}, \bar{\xi}) &= g(u, \bar{\xi}) + l_v \cdot g(\bar{\xi}, \bar{\xi}) \quad , \\ g(\bar{u}, \bar{u}) &= g(u + l_v \cdot \bar{\xi}, u + l_v \cdot \bar{\xi}) = \\ &= g(u, u) + 2 \cdot l_v \cdot g(u, \bar{\xi}) + l_v^2 \cdot g(\bar{\xi}, \bar{\xi}) \quad . \end{aligned} \quad (47)$$

The contravariant vector field $\bar{u} = \bar{u}^i \cdot \partial_i$ is usually interpreted as the velocity of an observer moving in a space-time described by a (\bar{L}_n, g) -space as its model. The contravariant vector u is a tangent vector field to the curve $x^i(\tau, r(\tau, \lambda) = r_0 = \text{const.}) = x^i(\tau, \lambda = \lambda_0 = \text{const.})$

$$\begin{aligned} u &= u^i \cdot \partial_i = \frac{\partial x^i}{\partial \tau} \cdot \partial_i \quad , \\ \bar{u} &= \frac{1}{g(u, u)} \cdot g(\bar{u}, u) \cdot u + \bar{g}[h_u(\bar{u})] \quad . \end{aligned} \quad (48)$$

The contravariant vector $\bar{\xi}$ is a collinear vector to the tangent vector ξ to the curve $x^i(\tau = \tau_0 = \text{const.}, r(\tau_0, \lambda)) = x^i(\tau = \tau_0 = \text{const.}, \lambda)$. This is so because of the relations

$$\bar{\xi} = \bar{\xi}^i \cdot \partial_i = \frac{\partial x^i}{\partial r} \cdot \partial_i \quad , \quad (49)$$

$$\frac{\partial x^i}{\partial r} = \frac{\partial x^i(\tau, r(\tau, \lambda))}{\partial r} = \frac{\partial x^i(\tau, \lambda(r, \tau))}{\partial r} = \frac{\partial x^i}{\partial \lambda} \cdot \frac{\partial \lambda}{\partial r} = \xi^i \cdot \frac{\partial \lambda}{\partial r} = \bar{\xi}^i \quad , \quad (50)$$

where

$$\begin{aligned} r &= r(\lambda, \tau) \quad , \quad \lambda = \lambda(\tau, r) \quad , \\ \bar{\xi} &= \bar{\xi}^i \cdot \partial_i = \frac{\partial \lambda}{\partial r} \cdot \xi^i \cdot \partial_i = \frac{\partial \lambda}{\partial r} \cdot \xi \quad , \quad \xi = \frac{\partial x^i}{\partial \lambda} \cdot \partial_i \quad . \end{aligned} \quad (51)$$

3. Further, since we wish to consider the vector field u as the velocity vector field of an observer moving at the curve $x^i(\tau, \lambda = \lambda_0 = \text{const.})$, interpreted as his world line, the vector field ξ (and $\bar{\xi}$ respectively) could be chosen to lie in the subspace orthogonal to u , i.e. u and $\bar{\xi}$ could obey the condition $g(u, \bar{\xi}) = 0$ and, therefore, $g(u, \xi) = 0$, $\xi = \xi_\perp = \bar{g}[h_u(\xi)]$, and $\bar{\xi} = \bar{\xi}_\perp$.

4. In the next step, we could consider the vector field $\bar{\xi}$ as a unit vector field in direction of the vector field ξ , i.e.

$$\bar{\xi}_\perp = n_\perp = \frac{\xi_\perp}{l_{\xi_\perp}} \quad , \quad g(u, n_\perp) = 0 \quad , \quad (52)$$

$$g(\bar{\xi}_\perp, \bar{\xi}_\perp) = g(n_\perp, n_\perp) = \frac{1}{l_{\xi_\perp}^2} \cdot g(\xi_\perp, \xi_\perp) = \mp \frac{1}{l_{\xi_\perp}^2} \cdot l_{\xi_\perp}^2 = \mp 1 \quad , \quad (53)$$

$$\begin{aligned} g(\bar{\xi}_\perp, \bar{\xi}_\perp) &= g\left(\frac{\partial \lambda}{\partial r} \cdot \xi_\perp, \frac{\partial \lambda}{\partial r} \cdot \xi_\perp\right) = \left(\frac{\partial \lambda}{\partial r}\right)^2 \cdot g(\xi_\perp, \xi_\perp) = \\ &= \mp \left(\frac{\partial \lambda}{\partial r}\right)^2 \cdot l_{\xi_\perp}^2 = \mp 1 \quad , \end{aligned} \quad (54)$$

$$\left(\frac{\partial \lambda}{\partial r}\right)^2 \cdot l_{\xi_\perp}^2 = 1 \quad , \quad l_{\xi_\perp}^2 = \left(\frac{\partial \lambda}{\partial r}\right)^{-2} \quad , \quad l_{\xi_\perp} = \pm \left(\frac{\partial \lambda}{\partial r}\right)^{-1} \quad . \quad (55)$$

After all above considerations for ξ_\perp and $\bar{\xi}_\perp$ we obtain the relations

$$\begin{aligned} g(\bar{u}, u) &= g(u, u) \quad , \\ g(\bar{u}, \bar{\xi}_\perp) &= l_v \cdot g(\bar{\xi}_\perp, \bar{\xi}_\perp) = l_v \cdot g(n_\perp, n_\perp) = \mp l_v \quad , \\ g(\bar{u}, \bar{u}) &= g(u, u) + l_v^2 \cdot g(n_\perp, n_\perp) = \\ &= \pm l_u^2 \mp l_v^2 = \frac{ds^2}{d\tau^2} \quad , \end{aligned} \quad (56)$$

$$\frac{ds^2}{d\tau^2} = g\left(\frac{d}{d\tau}, \frac{d}{d\tau}\right) = \pm l_u^2 \mp l_v^2 = \pm l_u^2 \cdot \left(1 - \frac{l_v^2}{l_u^2}\right) . \quad (57)$$

Moreover,

$$dx^i = (u^i + l_v \cdot n_\perp^i) \cdot d\tau + \frac{\partial r}{\partial \lambda} \cdot d\lambda \cdot n_\perp^i , \quad (58)$$

$$\begin{aligned} d &= d\tau \cdot (u + l_v \cdot n_\perp) + \frac{\partial r}{\partial \lambda} \cdot d\lambda \cdot n_\perp = \\ &= d\tau \cdot u + (d\tau \cdot l_v + \frac{\partial r}{\partial \lambda} \cdot d\lambda) \cdot n_\perp = \\ &= d\tau \cdot u + dr \cdot n_\perp , \end{aligned} \quad (59)$$

$$dr = d\tau \cdot l_v + \frac{\partial r}{\partial \lambda} \cdot d\lambda , \quad (60)$$

$$\frac{dr(\tau, \lambda)}{d\tau} = l_v , \quad \frac{d\lambda}{d\tau} = 0 , \quad (61)$$

$$\bar{u} = u + l_v \cdot n_\perp , \quad g(\bar{u}, u) = g(u, u) , \quad (62)$$

$$n_\perp = \bar{g}[h_u(n_\perp)] , \quad (63)$$

Remark. If $l_v \neq 0$ then the vector \bar{u} would have a part $l_v \cdot n_\perp$ orthogonal to u .

$$\begin{aligned} ds^2 &= g(d, d) = d\tau^2 \cdot g(u, u) + (d\tau \cdot l_v + \frac{\partial r}{\partial \lambda} \cdot d\lambda)^2 \cdot g(n_\perp, n_\perp) = \\ &= \pm d\tau^2 \cdot l_u^2 \mp (d\tau \cdot l_v + \frac{\partial r}{\partial \lambda} \cdot d\lambda)^2 = \pm d\tau^2 \cdot l_u^2 \mp d\tau^2 \cdot l_v^2 = \\ &= \pm d\tau^2 \cdot l_u^2 \mp dr^2 = \pm (l_u^2 \cdot d\tau^2 - dr^2) = \pm d\tau^2 \cdot (l_u^2 - \frac{dr^2}{d\tau^2}) = \\ &= \pm d\tau^2 \cdot (l_u^2 - l_v^2) , \end{aligned}$$

Therefore,

$$ds^2 = g(d, d) = \pm d\tau^2 \cdot (l_u^2 - l_v^2) , \quad (64)$$

$$\frac{ds^2}{d\tau^2} = g\left(\frac{d}{d\tau}, \frac{d}{d\tau}\right) = g(\bar{u}, \bar{u}) = \pm l_u^2 \cdot \left(1 - \frac{l_v^2}{l_u^2}\right) , \quad (65)$$

$$ds^2 = \pm l_u^2 \cdot d\tau^2 \cdot \left(1 - \frac{l_v^2}{l_u^2}\right) . \quad (66)$$

5. In the non-relativistic field theories the distance between two points $P \in M$ and $\bar{P} \in M$ is defined as

$$ds^2 = \mp dr^2 \quad (67)$$

and $l_u = 0$. This means that the distance between two neighboring points P and \bar{P} is the *space* distance measured between them in the rest (proper) reference frame of the observer (with absolute value l_u of his velocity u equal to zero). The time parameter τ is not considered as a co-ordinate in space-time, but as a parameter, independent of the frame of reference of the observer.

6. In the relativistic field theories and especially in the Einstein theory of gravitation dr is considered as the space distance between two neighboring points P and \bar{P} and $l_u \cdot d\tau$ is interpreted as the distance covered by a light signal in a time interval $d\tau$, measured by an observer in his proper frame of reference (when the observer in it is at rest). The quantity l_u is usually interpreted as the absolute value c of a light signal in vacuum, i.e. $l_u = c$, or l_u is normalized to 1, i.e. $l_u = 1$, if the proper time interval $d\tau$ is replaced with the proper distance interval $ds = c \cdot d\tau$, i.e. $\bar{u} = \frac{d}{d\tau}$ is replaced with $\bar{u}' = \frac{d}{c \cdot d\tau} = \frac{d}{ds}$, $ds = c \cdot d\tau$.

Therefore, there is a difference between the interpretation of the absolute value l_u of the velocity of an observer in classical and relativistic physics

(a) In classical physics, from the above consideration, it follows that $l_u = 0$ (the observer is at rest) and $ds = dr$ is the distance as space distance. The quantity l_v is the absolute value of the velocity between the observer at rest and a point \bar{P} in his neighborhood.

(b) In relativistic physics $l_u = c$, or $l_u = 1$, and l_u is not the absolute value of the velocity of the observer but the velocity of the light propagation which the observer could measure in his proper frame of reference. If we wish to interpret l_u as the absolute value of the velocity of the observer himself we should assume that $l_u \neq c$ or 1 (if the observer is not moving with the speed of light).

- There is the possibility to identify l_u with l_v as the absolute value of the velocity of the observer at a point P at his world line, measured with respect to a neighboring point \bar{P} with the same proper time as the point P . Under this assumption, the ordinary differential becomes a null (isotropic) vector field [$g(d, d) = 0$, $l_d = 0$, $l_u = l_v \neq 0$] in the proper frame of reference of the observer.
- We could also interpret l_u as the absolute value of the velocity of the observer with respect to another frame of reference or
- we could consider l_u as the absolute value of the velocity of a signal coming to the observer with velocity, different from the velocity of light. On the basis of the last assumption we can describe the propagation of signals with propagation velocity different from the velocity of light (for instance, the propagation of sound signals or (may be) gravitational signals).

If $l_v = 0$ then $\bar{u} = u$ and u could be

- the velocity vector field $u = \frac{d}{d\tau}$ of an observer ($l_u \neq 0$, $u = l_u \cdot n_{||}$) in his proper frame of reference along his world line. Since in his proper frame of reference the observer is at rest, u could be interpreted as the velocity of a clock measuring the length (proper time) of the world line by the use of the parameter τ or
- the velocity of a signal detected or emitted by the observer.

There is another way for considering the ordinary differential as a contravariant vector field with its relations to the notions of velocity and of space velocity.

3.1.2 Distance in a (\bar{L}_n, g) -space for world and space lines depending to each other

1. Let us consider the ordinary differential $d = dx^i \cdot \partial_i$ as a contravariant vector field over a two parameter congruence $x^i = x^i(\tau, \lambda)$. Then

$$dx^i = \frac{\partial x^i(\tau, \lambda)}{\partial \tau} \cdot d\tau + \frac{\partial x^i(\tau, \lambda)}{\partial \lambda} \cdot d\lambda = u^i \cdot d\tau + \xi_\perp^i \cdot d\lambda , \quad (68)$$

where

$$u^i = \frac{\partial x^i(\tau, \lambda)}{\partial \tau} , \quad \xi_\perp^i = \frac{\partial x^i(\tau, \lambda)}{\partial \lambda} . \quad (69)$$

The ordinary differential will have the form

$$d = dx^i \cdot \partial_i = d\tau \cdot u + d\lambda \cdot \xi_\perp , \quad u = u^i \cdot \partial_i , \quad \xi_\perp = \xi_\perp^i \cdot \partial_i . \quad (70)$$

If we impose the additional condition

$$g(u, \xi_\perp) = 0 \quad (71)$$

we will have the relations

$$g(d, u) = d\tau \cdot g(u, u) = e \cdot d\tau = \pm l_u^2 \cdot d\tau , \quad (72)$$

$$g(\xi_\perp, d) = d\lambda \cdot g(\xi_\perp, \xi_\perp) = \mp l_{\xi_\perp}^2 \cdot d\lambda . \quad (73)$$

2. On the other side, if we consider the projections of d collinear and orthogonal to u we obtain the relations

$$d = \frac{1}{e} \cdot g(u, d) \cdot u + \bar{g}[h_u(d)] = d_\parallel + d_\perp , \quad (74)$$

where

$$d_\parallel = \frac{1}{e} \cdot g(u, d) \cdot u , \quad d_\perp = \bar{g}[h_u(d)] . \quad (75)$$

For the explicit form of d as $d = d\tau \cdot u + d\lambda \cdot \xi_\perp$ we have [under the condition $g(u, \xi_\perp) = 0$]

$$d_\parallel = \frac{1}{e} \cdot g(u, d) \cdot u = \frac{1}{e} \cdot e \cdot d\tau \cdot u = d\tau \cdot u , \quad (76)$$

$$\begin{aligned} d_\perp &= \bar{g}[h_u(d)] = \bar{g}[h_u(d\tau \cdot u + d\lambda \cdot \xi_\perp)] = \\ &= \bar{g}[h_u(d\lambda \cdot \xi_\perp)] = d\lambda \cdot \bar{g}[h_u(\xi_\perp)] = d\lambda \cdot \xi_\perp . \end{aligned} \quad (77)$$

Therefore, the representation of d as $d = d\tau \cdot u + d\lambda \cdot \xi_\perp$ is the representation in its collinear and orthogonal to u parts.

3. Since

$$\begin{aligned} g(d, d) &= ds^2 = g(d\tau \cdot u + d\lambda \cdot \xi_\perp, d\tau \cdot u + d\lambda \cdot \xi_\perp) = \\ &= d\tau^2 \cdot g(u, u) + d\lambda^2 \cdot g(\xi_\perp, \xi_\perp) = \\ &= \pm l_u^2 \cdot d\tau^2 \mp l_{\xi_\perp}^2 \cdot d\lambda^2 , \end{aligned} \quad (78)$$

it follows for the line element ds

$$ds^2 = \pm l_u^2 \cdot d\tau^2 \mp l_{\xi_\perp}^2 \cdot d\lambda^2 = \pm l_u^2 \cdot d\tau^2 \cdot \left(1 - \frac{l_{\xi_\perp}^2 \cdot d\lambda^2}{l_u^2 \cdot d\tau^2}\right) . \quad (79)$$

If we chose the contravariant non-isotropic (non-null) vector field ξ_\perp as a unit vector field, i.e. if $l_{\xi_\perp} = 1$, $\xi_\perp = \tilde{n}_\perp$, $g(\tilde{n}_\perp, \tilde{n}_\perp) = \mp 1 = \mp l_{\xi_\perp}^2$, then

$$ds^2 = \pm l_u^2 \cdot d\tau^2 \cdot \left(1 - \frac{1}{l_u^2} \cdot \frac{d\lambda^2}{d\tau^2}\right) \quad . \quad (80)$$

If we, further, interpret $d\lambda$ as a distance along a curve with a tangent vector ξ_\perp , orthogonal to u , we can define and interpret the expression

$$\frac{d\lambda}{d\tau} = l_v \quad (81)$$

as the 3-dimensional space velocity of a material point along the curve $x^i(\tau, \lambda)$ with tangential vector $\xi_\perp = \tilde{n}_\perp$ along the curve $x^i(\tau_0, \lambda)$. Then

$$ds^2 = \pm l_u^2 \cdot d\tau^2 \cdot \left(1 - \frac{l_v^2}{l_u^2}\right) \quad . \quad (82)$$

Remark. Here, it is assumed that the parameter λ is depending on the parameter τ , i.e. $\lambda = \lambda(\tau)$, $\tau = \tau(\lambda)$, and $d\lambda/d\tau \neq 0$. In the opposite case, where λ and τ are parameters independent to each other, $d\lambda/d\tau = 0$.

The quantity l_u is interpreted as the absolute value of the velocity of a signal (in the relativity theory it is interpreted as the absolute value of the velocity of light in vacuum). The parameter τ is interpreted as the proper time of an observer moving on a trajectory $x^i(\tau, \lambda_0)$ interpreted as his world line $x^i(\tau, \lambda_0)$. The quantity l_v is interpreted as the absolute value of the velocity of a material point moving along a space distance λ from the trajectory of the observer $x^i(\tau, \lambda_0)$.

4. If we now turn back to the general case, when $l_{\xi_\perp} \neq 1$, $\xi_\perp \neq \tilde{n}_\perp$, we have the relation

$$ds^2 = \pm l_u^2 \cdot d\tau^2 \cdot \left[1 - \frac{1}{l_u^2} \cdot \left(\frac{l_{\xi_\perp}^2}{d\tau^2} \cdot d\lambda^2\right)\right] \quad . \quad (83)$$

Then we can introduce the abbreviation

$$dl^2 := l_{\xi_\perp}^2 \cdot d\lambda^2 \quad , \quad (84)$$

interpreted as the square of the *space distance* of a point in $n - 1$ -dimensional subspace of M ($\dim M = n$), ($n = 4$) from the trajectory (world line) of the observer with proper time τ and tangential vector u , orthogonal to $\xi_\perp = l_{\xi_\perp} \cdot \tilde{n}_\perp$. The square ds^2 of the distance ds in the n -dimensional manifold M will have the form

$$ds^2 = \pm l_u^2 \cdot d\tau^2 \cdot \left[1 - \frac{1}{l_u^2} \cdot \frac{dl^2}{d\tau^2}\right] \quad . \quad (85)$$

If we again denote

$$\frac{dl^2}{d\tau^2} := \bar{l}_v^2 \quad (86)$$

we obtain

$$ds^2 = \pm l_u^2 \cdot d\tau^2 \cdot \left(1 - \frac{\bar{l}_v^2}{l_u^2}\right) \quad , \quad (87)$$

where \bar{l}_v could be interpreted again as the absolute value of the space velocity of a point moving at a space distance dl from the world line $x^i(\tau, \lambda_0)$ of the

observer. In this general case, the parameter λ is not interpreted as a space distance. Instead of $d\lambda$ the quantity $dl = l_{\xi_\perp} \cdot d\lambda$ has this interpretation.

In general, we do not need to search for interpretation of an expression as a space velocity in the above considerations if we consider only the structure of the square ds^2 of the space-time distance ds

$$ds^2 = \pm(l_u^2 \cdot d\tau^2 - l_{\xi_\perp}^2 \cdot d\lambda^2) = \pm(l_u^2 \cdot d\tau^2 - dl^2) . \quad (88)$$

If a signal with absolute value l_u of its velocity is covering a space distance dl with $dl^2 = l_{\xi_\perp}^2 \cdot d\lambda^2$ in the proper time interval $d\tau$ of the observer then $ds^2 = g(d, d) = 0$ and the ordinary differential becomes a null (isotropic) vector field, where

$$ds^2 = 0 \quad , \quad l_u^2 \cdot d\tau^2 = l_{\xi_\perp}^2 \cdot d\lambda^2 = dl^2 \quad , \quad (89)$$

$$dl = \pm l_{\xi_\perp} \cdot d\lambda \quad , \quad l_u \cdot d\tau = \pm l_{\xi_\perp} \cdot d\lambda = dl \quad , \quad (90)$$

$$d\tau = \frac{dl}{l_u} = \pm \frac{l_{\xi_\perp}}{l_u} \cdot d\lambda \quad , \quad l_{\xi_\perp} \cdot d\lambda = \pm dl \quad , \quad (91)$$

$$l_u > 0 \quad , \quad l_{\xi_\perp} > 0 \quad .$$

3.2 Measuring a distance in (\bar{L}_n, g) -spaces

A. If the notion of distance is introduced in a space-time, modeled by a (\bar{L}_n, g) -space, we have to decide *what is the meaning of the vector field u as tangent vector to a trajectory interpreted as the world line of an observer*. On the basis of the above considerations, we have four possible interpretations for the meaning of the vector field u as

1. Velocity vector field of a propagating signal in space-time identified with the tangent vector field u at the world line of an observer. The signal is detected or emitted by the observer on his world line and the absolute value l_u of u is identified with the absolute value of the velocity of the signal in- or outcoming to the observer.

2. Velocity vector field of an observer moving in space-time. In this case $l_u \neq 0$ and the space-time should have a definite metric, i.e. $Sgn g = \pm n$, $dim M = n$ (for instance, motion of an observer in an Euclidean space considered as a model of space-time). The observer, moving in space-time, could consider processes happened in its subspace orthogonal to his velocity. The observer will move in a flow and consider the characteristics of the flow from his own frame of reference.

3. Velocity of a clock moving in space-time and determining the proper time by a periodical process in the frame of reference of an observer. The velocity u of the periodical process in the clock in space-time is with fixed absolute value l_u , i.e. $l_u = \text{const}$. The time interval $d\tau$ measured by the clock corresponds to the length ds of its world line, i.e. $d\tau^2 = \pm \text{const} \cdot ds^2$. Under the assumption for the constant velocity of the periodical process in the clock, we consider the periodical process as indicator for the time interval $d\tau$ in the proper frame of reference of the clock and of the observer respectively.

4. Velocity u of an $(n-1)$ -dimensional subspace moving in time with $l_u \neq 0$. If the subspace deforms in some way, the deformations reflect on the kinematic characteristics of the vector field u and u is used as an indicator for the changing properties of the subspace, considered as the space of an observer (laboratory)

where a physical system is investigated. This type of interpretation requires not only the existence of the velocity vector field u with $l_u \neq 0$ but also the existence of (at least one) orthogonal to u vector field ξ_\perp , $g(u, \xi_\perp) = 0$, lying in the orthogonal to u subspace $T^{\perp u}(M)$.

All indicated interpretations could be used in solving different physical problems related to motions of physical systems in space-time.

B. After introducing the notion of distance, the question arises *how a space distance between two points in a space could be measured*. We could distinguish three types of measurements:

1. Direct measurements by using a measuring device (e.g. a roulette, a linear (running) meter, yard-measure-stick etc.)
2. Direct measurements by sending signals from a basic point to a fixed point of space and detecting at the basic point the reflected by the fixed point signal.
3. Indirect measurement by receiving signals from a fixed point of space without sending a signal to it.

Let us now consider every type of measurements more closely.

1. *Direct measurements by using a measuring device*. The space distance between two points A and B in a space could be measured by a second observer moving from point A (where the first observer is at rest) to point B in space. At the same time, the second observer moves in time from point B to point B' . The space distance measured by the observer with world line AA' could be denoted as $\Delta r = AB$ and the time period passed as $\Delta \tau = AA'$. This is a direct measurement of the space distance $AB = \Delta r$ from point A to point B in the space during the time $AA' = \Delta \tau$. It is *assumed* that point A and point B are at rest during the measurement. Instead of measuring the space distance AB the observers measure the space distance $A'B'$ which exists at the time $\tau + \Delta \tau$ if the measurement has began at the time τ from the point of the first observer with world line AA' .

2. *Direct measurements by sending signals from a basic point to a fixed point of space and detecting at the basic point the reflected by the fixed point signal*. The space distance between two points A and B in a space could be measured by sending a signal with velocity with absolute value $l_u \neq 0$. Then AB of the curve $x^i(\tau = \tau_0, r)$ through point B is the distance Δr at the time $\tau(A) = \tau_0$ and $\tau(B) = \tau_0$.

$A'B'$ of the curve $x^i(\tau = \tau_0 + \Delta \tau, r)$ is the space distance $\Delta r'$ at the time $\tau(A') = \tau_1$. At this time the signal is received at point B' which is point B at the time τ_1 , i.e. $\tau(B') = \tau_1$. $B'A'$ is the space distance between B and A at the time $\tau(A') = \tau_1$, where $\tau(B') = \tau_1$, $\tau(B'') = \tau_2$. At the time (τ_2) the point $B(\tau_0)$ will be moved in the time to point $B''(\tau_2)$. The signal will be propagated

(a) for the time interval $AA' = \tau_1 - \tau_0$ to the point B' at the time τ_1 at the space distance $\Delta r = l_u \cdot (\tau_1 - \tau_0)$, where l_u is the velocity of the signal measured by the observer with world line AA' .

(b) for the time interval $A'A''$ from point B' at the time τ_1 to the point A'' at the time τ_2 at a space distance $l_u \cdot (\tau_2 - \tau_1)$. The whole space distance covered by the signal in the time interval $AA'A'' = \Delta \tau = \tau_2 - \tau_0$ is $l_u \cdot (\tau_2 - \tau_0) = l_u \cdot (\tau_2 - \tau_1) + l_u \cdot (\tau_1 - \tau_0)$.

If we now *assume* that point A and point B are at rest to each other and

the space distance between them does not change in the time then

$$l_u \cdot (\tau_2 - \tau_1) = l_u \cdot (\tau_1 - \tau_0) \quad (92)$$

and

$$l_u \cdot (\tau_2 - \tau_0) = 2 \cdot l_u \cdot (\tau_1 - \tau_0) = 2 \cdot A'B'(\tau_1) = 2 \cdot AB(\tau_0) \quad (93)$$

Therefore, the space distance between point A and point B (at any time, if both the points are at rest to each other) is

$$AB = \frac{1}{2} \cdot l_u \cdot (\tau_2 - \tau_0) \quad (94)$$

where $\Delta \tau = \tau_2 - \tau_0$ is the time interval for the propagation of a signal from point A to point B and from point B back to point A .

3. *Indirect measurement by receiving signals from a fixed point of space without sending a signal to it.* If the space distance between point A and point B is changing in the time and at point B there is an emitter then the frequency of the emitter could change in the time related to the centrifugal (centripetal) or Coriolis' velocities and accelerations between both the points A and B . Therefore, a criteria for no relative motion between two (space) points (points with one and the same proper time) could be the lack of change of the frequency of the signals emitted from the second point B to the basic point A . [But there could be motions of an emitter which could so change its frequency that the changes compensate each other and the observer at the basic point A could come to the conclusion that there is no motions between points A and B .]

If an emitter at point $B(\tau_0)$ emits a signal with velocity \bar{u} and frequency $\bar{\omega}$ then this signal will be received (detected) at the point $A'(\tau_1)$ after a time interval $AA' = \Delta \tau = \tau_1 - \tau_0$ by an observer (detector) moving in the time interval $\Delta \tau$ from point $A(\tau_0)$ to point $A'(\tau_1)$ on his world line $x^i(\tau)$. If the emitter is moving relatively to point A with relative velocity $_{rel}v$ and / or with relative acceleration $_{rel}a$ then the detected at the point A' frequency ω will differ from the emitted frequency $\bar{\omega}$. If both the points A and B are at rest to each other then $\bar{\omega} = \omega$.

C. The question arises *how can we find the space distance between two points A and B lying in such a way in the space that only signals emitted from the one point (point B) could be detected at the basic point (point A), where an observer detects the signal from point B .* First of all, if we knew the propagation velocity l_u of a signal and the difference $\bar{\omega} - \omega$ between the emitted frequency $\bar{\omega}$ and the detected frequency ω we can try to find out the relative velocity (and acceleration) between the emitter (at a point B) and the observer (at a point A). For doing that we will need relations between the difference $\bar{\omega} - \omega$ and the relative velocity (and acceleration) between both the points. Such relations could be found on the basis of the structures of the relative velocity and the relative acceleration and their decompositions in centrifugal (centripetal) relative velocity and relative acceleration and Coriolis relative velocity and relative acceleration [12].

4 Kinematic effects related to the relative velocity and to the relative acceleration

1. Let us now consider the change of a null vector field \tilde{k} under the influence of the relative velocity and of the relative acceleration on the corresponding emitter and its frequency with respect to an observer detecting the emitted radiation by the emitter.

Let \bar{k}_\perp be the orthogonal to u part of the null vector field \bar{k} corresponding to the null vector field \tilde{k} after the influence of the relative velocity ${}_{rel}v$ and / or the relative acceleration ${}_{rel}a$

$$\bar{k} = \tilde{k} + {}_{rel}k, \quad \bar{k} = \bar{k}_\parallel + \bar{k}_\perp, \quad \tilde{k} = k_\parallel + k_\perp, \quad (95)$$

$${}_{rel}k = {}_{rel}k_\parallel + {}_{rel}k_\perp, \quad (96)$$

$$\bar{k}_\perp = k_\perp + {}_{rel}k_\perp, \quad (97)$$

where ${}_{rel}k$ could depend on the relative velocity ${}_{rel}v$ ([3] - Ch. 10) and on the relative acceleration ${}_{rel}a$ ([3] - Ch. 11, Ch. 12).

If $\bar{k} = \tilde{k} + {}_{rel}k$ then

$$\begin{aligned} g(\bar{k}, \bar{k}) &= 0 = g({}_{rel}k, {}_{rel}k) + 2 \cdot g(\tilde{k}, {}_{rel}k), \\ g({}_{rel}k, {}_{rel}k) &= -2 \cdot g(\tilde{k}, {}_{rel}k). \end{aligned} \quad (98)$$

In the previous sections we have considered the representation of \tilde{k} as $\tilde{k} = k_\parallel + k_\perp$, where

$$k_\parallel = \pm l_{k_\parallel} \cdot n_\parallel = \pm \frac{\omega}{l_u} \cdot n_\parallel, \quad l_{k_\parallel} = \frac{\omega}{l_u}, \quad (99)$$

$$k_\perp = \mp l_{k_\perp} \cdot \tilde{n}_\perp = \mp \frac{\omega}{l_u} \cdot \tilde{n}_\perp, \quad l_{k_\perp} = \frac{\omega}{l_u} = l_{k_\parallel}. \quad (100)$$

The unit vector \tilde{n}_\perp is orthogonal to the vector u , i.e. $g(u, \tilde{n}_\perp) = 0$ because of $g(u, k_\perp) = \mp l_{k_\perp} \cdot g(u, \tilde{n}_\perp) = 0$, $l_{k_\perp} \neq 0$, $l_u \neq 0$. Therefore, $g(\tilde{n}_\perp, \tilde{n}_\perp) = \mp 1 = \mp l_{\tilde{n}_\perp}^2$, $l_{\tilde{n}_\perp} \neq 0$.

We can represent the unit vector \tilde{n}_\perp (orthogonal to u) in two parts: one part n_\perp collinear to the vector field ξ_\perp (orthogonal to u) and one part m_\perp orthogonal to the vectors u and ξ_\perp , i.e.

$$\tilde{n}_\perp = \alpha \cdot n_\perp + \beta \cdot m_\perp, \quad (101)$$

$$g(\tilde{n}_\perp, \tilde{n}_\perp) = \mp 1 = \mp l_{\tilde{n}_\perp}^2, \quad l_{\tilde{n}_\perp} > 0, \quad l_{\tilde{n}_\perp} = 1, \quad (102)$$

$$\eta_\perp : = l_{\xi_\perp} \cdot m_\perp, \quad m_\perp = \frac{\eta_\perp}{l_{\xi_\perp}}, \quad g(m_\perp, m_\perp) = \mp 1 \quad (103)$$

$$g(\eta_\perp, \eta_\perp) = l_{\xi_\perp}^2 \cdot g(m_\perp, m_\perp) = \mp l_{\xi_\perp}^2 = g(\xi_\perp, \xi_\perp), \quad (104)$$

$$g(\eta_\perp, \xi_\perp) = 0, \quad g(m_\perp, n_\perp) = 0, \quad (105)$$

where

$$n_\perp = \frac{\xi_\perp}{l_{\xi_\perp}}, \quad g(n_\perp, u) = 0, \quad g(n_\perp, n_\perp) = \mp 1 = \mp l_{n_\perp}^2, \quad (106)$$

$$l_{n_\perp} > 0, \quad l_{n_\perp} = 1 \quad (107)$$

$$m_\perp = \frac{\eta_\perp}{l_{\xi_\perp}} = \frac{v_c}{l_{v_c}}, \quad g(m_\perp, u) = 0, \quad g(m_\perp, \xi_\perp) = 0. \quad (108)$$

The vector field v_c is the Coriolis velocity vector field [12] orthogonal to u and to the centrifugal (centripetal) velocity v_z collinear to ξ_\perp . Since

$$v_c = \mp l_{v_c} \cdot m_\perp \quad , \quad g(v_c, v_c) = \mp l_{v_c}^2 \quad , \quad (109)$$

we also have

$$g(m_\perp, m_\perp) = \mp 1 = \mp l_{m_\perp}^2 \quad , \quad l_{m_\perp} > 0 \quad , \quad l_{m_\perp} = 1 \quad . \quad (110)$$

The Coriolis velocity v_c is related to the change of the vector ξ_\perp in direction orthogonal to u and ξ_\perp .

Since \tilde{n}_\perp is a unit vector as well as the vectors n_\perp and m_\perp , and, further, $g(n_\perp, m_\perp) = 0$, we obtain

$$g(\tilde{n}_\perp, \tilde{n}_\perp) = \mp 1 = g(\alpha \cdot n_\perp + \beta \cdot m_\perp, \alpha \cdot n_\perp + \beta \cdot m_\perp) = \quad (111)$$

$$= \alpha^2 \cdot g(n_\perp, n_\perp) + \beta^2 \cdot g(m_\perp, m_\perp) = \quad (112)$$

$$= \mp \alpha^2 \mp \beta^2 \quad . \quad (113)$$

Therefore,

$$\alpha^2 + \beta^2 = 1 \quad . \quad (114)$$

On the other side,

$$\begin{aligned} g(\tilde{n}_\perp, n_\perp) &= g(\alpha \cdot n_\perp + \beta \cdot m_\perp, n_\perp) = \\ &= \alpha \cdot g(n_\perp, n_\perp) = \mp \alpha \quad , \end{aligned} \quad (115)$$

$$\begin{aligned} g(\tilde{n}_\perp, m_\perp) &= g(\alpha \cdot n_\perp + \beta \cdot m_\perp, m_\perp) = \\ &= \beta \cdot g(n_\perp, m_\perp) = \mp \beta \quad . \end{aligned} \quad (116)$$

i.e.

$$\alpha = \mp g(\tilde{n}_\perp, n_\perp) = \mp l_{\tilde{n}_\perp} \cdot l_{n_\perp} \cdot \cos(\tilde{n}_\perp, n_\perp) = \mp \cos(\tilde{n}_\perp, n_\perp) \quad , \quad (117)$$

$$\beta = \mp g(\tilde{n}_\perp, m_\perp) = \mp l_{\tilde{n}_\perp} \cdot l_{m_\perp} \cdot \cos(\tilde{n}_\perp, m_\perp) = \mp \cos(\tilde{n}_\perp, m_\perp) \quad . \quad (118)$$

Therefore, α and β appear as direction cosines of n_\perp and m_\perp with respect to the unit vector \tilde{n}_\perp . Since

$$\cos^2(\tilde{n}_\perp, n_\perp) + \cos^2(\tilde{n}_\perp, m_\perp) = 1 \quad , \quad (119)$$

it follows that

$$\cos^2(\tilde{n}_\perp, m_\perp) = 1 - \cos^2(\tilde{n}_\perp, n_\perp) = 1 - \sin^2(\tilde{n}_\perp, m_\perp) = \sin^2(\tilde{n}_\perp, n_\perp) \quad ,$$

$$\sin^2(\tilde{n}_\perp, m_\perp) = \cos^2(\tilde{n}_\perp, n_\perp) \quad ,$$

$$\cos(\tilde{n}_\perp, m_\perp) = \pm \sin(\tilde{n}_\perp, n_\perp) \quad ,$$

$$\alpha = \mp \cos(\tilde{n}_\perp, n_\perp) \quad , \quad (120)$$

$$\beta = \mp \sin(\tilde{n}_\perp, n_\perp) \quad , \quad (121)$$

$$\begin{aligned} \tilde{n}_\perp &= \alpha \cdot n_\perp + \beta \cdot m_\perp = \\ &= \mp [\cos(\tilde{n}_\perp, n_\perp) \cdot n_\perp + \sin(\tilde{n}_\perp, n_\perp) \cdot m_\perp] \quad . \end{aligned} \quad (122)$$

If we denote the angle $(\tilde{n}_\perp, n_\perp)$ between the vectors \tilde{n}_\perp and n_\perp as $\theta = (\tilde{n}_\perp, n_\perp)$ then the above relations could be written in the form

$$\begin{aligned} \cos^2(\tilde{n}_\perp, m_\perp) &= \sin^2\theta , \\ \sin^2(\tilde{n}_\perp, m_\perp) &= \cos^2\theta , \\ \alpha &= \mp \cos\theta , \\ \beta &= \mp \sin\theta , \end{aligned} \quad (123)$$

$$\beta = \mp \sin\theta , \quad (124)$$

$$\tilde{n}_\perp = \alpha \cdot n_\perp + \beta \cdot m_\perp = \mp [\cos\theta \cdot n_\perp + \sin\theta \cdot m_\perp] . \quad (125)$$

Further, since $k_\perp = \mp l_{k_\perp} \cdot \tilde{n}_\perp$ then (see above)

$$g(k_\perp, k_\perp) = l_{k_\perp}^2 \cdot g(\tilde{n}_\perp, \tilde{n}_\perp) = \mp l_{k_\perp}^2 , \quad g(\tilde{n}_\perp, \tilde{n}_\perp) = \mp 1 , \quad (126)$$

$$k_\perp = \mp \frac{\omega}{l_u} \cdot \tilde{n}_\perp , \quad k_\parallel = \pm \frac{\omega}{l_u} \cdot n_\parallel , \quad l_{k_\perp} = \frac{\omega}{l_u} = l_{k_\parallel} , \quad (127)$$

$$g(\tilde{n}_\perp, k_\perp) = \mp l_{k_\perp} \cdot g(\tilde{n}_\perp, \tilde{n}_\perp) = \frac{\omega}{l_u} = l_{k_\parallel} = l_{k_\perp} . \quad (128)$$

2. For the contravariant null vector field \bar{k} we have analogous relations as for the contravariant null vector field \tilde{k} (just changing \tilde{k} with \bar{k} , ω with $\bar{\omega}$, and \tilde{n}_\perp with \tilde{n}'_\perp)

$$\bar{k} = \bar{k}_\parallel + \bar{k}_\perp , \quad \bar{\omega} = g(u, \bar{k}) , \quad (129)$$

$$\bar{k}_\parallel = \pm \frac{\bar{\omega}}{l_u} \cdot n_\parallel , \quad l_{\bar{k}_\parallel} = \frac{\bar{\omega}}{l_u} , \quad (130)$$

$$\bar{k}_\perp = \mp \frac{\bar{\omega}}{l_u} \cdot \tilde{n}'_\perp , \quad l_{\bar{k}_\perp} = \frac{\bar{\omega}}{l_u} = l_{\bar{k}_\parallel} , \quad (131)$$

$$g(\bar{k}_\perp, \bar{k}_\perp) = \frac{\bar{\omega}^2}{l_u^2} \cdot g(\tilde{n}'_\perp, \tilde{n}'_\perp) = \mp l_{\bar{k}_\perp}^2 = \mp \frac{\bar{\omega}^2}{l_u^2} , \quad (132)$$

$$g(\tilde{n}'_\perp, \tilde{n}'_\perp) = \mp 1 \quad (133)$$

$$g(\tilde{n}'_\perp, \bar{k}_\perp) = \mp l_{\bar{k}_\perp} \cdot g(\tilde{n}'_\perp, \tilde{n}'_\perp) = \frac{\bar{\omega}}{l_u} = l_{\bar{k}_\parallel} = l_{\bar{k}_\perp} . \quad (134)$$

From $\bar{k} = \tilde{k} + {}_{rel}k$, $\bar{k}_\perp = \tilde{k}_\perp + {}_{rel}k_\perp$, and

$$\mp \frac{\bar{\omega}}{l_u} \cdot \tilde{n}'_\perp = \mp \frac{\omega}{l_u} \cdot \tilde{n}_\perp + {}_{rel}k_\perp , \quad (135)$$

it follows that

$$\begin{aligned} g(\tilde{n}_\perp, \bar{k}) &= g(\tilde{n}_\perp, k) + g(\tilde{n}_\perp, {}_{rel}k) , \\ g(\tilde{n}_\perp, \bar{k}_\perp) &= g(\tilde{n}_\perp, k_\perp) + g(\tilde{n}_\perp, {}_{rel}k_\perp) , \end{aligned} \quad (136)$$

$$\mp \frac{\bar{\omega}}{l_u} \cdot g(\tilde{n}'_\perp, \tilde{n}_\perp) = \frac{\omega}{l_u} + g({}_{rel}k_\perp, \tilde{n}_\perp) . \quad (137)$$

The vector \tilde{n}'_\perp could be represented by the use of the vectors n_\perp and m_\perp in the form

$$\tilde{n}'_\perp = \alpha' \cdot n_\perp + \beta' \cdot m_\perp , \quad (138)$$

where

$$\begin{aligned}
g(\tilde{n}'_{\perp}, \tilde{n}'_{\perp}) &= \mp 1 = g(\alpha' \cdot n_{\perp} + \beta' \cdot m_{\perp}, \alpha' \cdot n_{\perp} + \beta' \cdot m_{\perp}) = \\
&= \alpha'^2 \cdot g(n_{\perp}, n_{\perp}) + \beta'^2 \cdot g(m_{\perp}, m_{\perp}) = \\
&= \mp \alpha'^2 \mp \beta'^2 .
\end{aligned} \tag{139}$$

Therefore,

$$\alpha'^2 + \beta'^2 = 1 . \tag{140}$$

On the other side we have analogous relations as in the case of the vector \tilde{n}_{\perp} :

$$\begin{aligned}
g(\tilde{n}'_{\perp}, n_{\perp}) &= g(\alpha' \cdot n_{\perp} + \beta' \cdot m_{\perp}, n_{\perp}) = \\
&= \alpha' \cdot g(n_{\perp}, n_{\perp}) = \mp \alpha' ,
\end{aligned} \tag{141}$$

$$\begin{aligned}
g(\tilde{n}'_{\perp}, m_{\perp}) &= g(\alpha' \cdot n_{\perp} + \beta' \cdot m_{\perp}, m_{\perp}) = \\
&= \beta' \cdot g(m_{\perp}, m_{\perp}) = \mp \beta' .
\end{aligned} \tag{142}$$

i.e.

$$\alpha' = \mp g(\tilde{n}'_{\perp}, n_{\perp}) = \mp l_{\tilde{n}'_{\perp}} \cdot l_{n_{\perp}} \cdot \cos(\tilde{n}'_{\perp}, n_{\perp}) = \mp \cos(\tilde{n}'_{\perp}, n_{\perp}) , \tag{143}$$

$$\beta' = \mp g(\tilde{n}'_{\perp}, m_{\perp}) = \mp l_{\tilde{n}'_{\perp}} \cdot l_{m_{\perp}} \cdot \cos(\tilde{n}'_{\perp}, m_{\perp}) = \mp \cos(\tilde{n}'_{\perp}, m_{\perp}) . \tag{144}$$

Therefore, α' and β' appear as direction cosines of n_{\perp} and m_{\perp} with respect to the unit vector \tilde{n}'_{\perp} . Since

$$\cos^2(\tilde{n}'_{\perp}, n_{\perp}) + \cos^2(\tilde{n}'_{\perp}, m_{\perp}) = 1 , \tag{145}$$

it follows that

$$\cos^2(\tilde{n}'_{\perp}, m_{\perp}) = 1 - \cos^2(\tilde{n}'_{\perp}, n_{\perp}) = \sin^2(\tilde{n}'_{\perp}, m_{\perp}) = \sin^2(\tilde{n}'_{\perp}, n_{\perp}) ,$$

$$\sin^2(\tilde{n}'_{\perp}, m_{\perp}) = \cos^2(\tilde{n}'_{\perp}, n_{\perp}) ,$$

$$\cos(\tilde{n}'_{\perp}, m_{\perp}) = \pm \sin(\tilde{n}'_{\perp}, n_{\perp}) ,$$

$$\alpha' = \mp \cos(\tilde{n}'_{\perp}, n_{\perp}) , \tag{146}$$

$$\beta' = \mp \sin(\tilde{n}'_{\perp}, n_{\perp}) , \tag{147}$$

$$\begin{aligned}
\tilde{n}'_{\perp} &= \alpha' \cdot n_{\perp} + \beta' \cdot m_{\perp} = \\
&= \mp [\cos(\tilde{n}'_{\perp}, n_{\perp}) \cdot n_{\perp} + \sin(\tilde{n}'_{\perp}, n_{\perp}) \cdot m_{\perp}] .
\end{aligned} \tag{148}$$

If we denote the angle $(\tilde{n}'_{\perp}, n_{\perp})$ between the vectors \tilde{n}'_{\perp} and n_{\perp} as $\theta' = (\tilde{n}'_{\perp}, n_{\perp})$ then the above relations could be written in the form

$$\cos^2(\tilde{n}'_{\perp}, m_{\perp}) = \sin^2 \theta' ,$$

$$\sin^2(\tilde{n}'_{\perp}, m_{\perp}) = \cos^2 \theta' ,$$

$$\alpha' = \mp \cos \theta' , \tag{149}$$

$$\beta' = \mp \sin \theta' , \tag{150}$$

$$\tilde{n}_{\perp} = \alpha \cdot n_{\perp} + \beta \cdot m_{\perp} = \mp [\cos \theta \cdot n_{\perp} + \sin \theta \cdot m_{\perp}] . \tag{151}$$

From the relation

$$\mp \frac{\bar{\omega}}{l_u} \cdot \tilde{n}'_{\perp} = \mp \frac{\omega}{l_u} \cdot \tilde{n}_{\perp} + {}_{rel}k_{\perp} \quad (152)$$

the relations for ${}_{rel}k_{\perp}$ follow

$$\mp \frac{\bar{\omega}}{l_u} \cdot g(\tilde{n}'_{\perp}, n_{\perp}) = \mp \frac{\omega}{l_u} \cdot g(\tilde{n}_{\perp}, n_{\perp}) + g({}_{rel}k_{\perp}, n_{\perp}) \quad , \quad (153)$$

$$\mp \alpha' \cdot \frac{\bar{\omega}}{l_u} = \mp \alpha \cdot \frac{\omega}{l_u} + g({}_{rel}k_{\perp}, n_{\perp}) \quad , \quad (154)$$

$$\frac{\bar{\omega}}{l_u} \cdot \cos \theta' = \frac{\omega}{l_u} \cdot \cos \theta + g({}_{rel}k_{\perp}, n_{\perp}) \quad , \quad (155)$$

$$\bar{\omega} \cdot \cos \theta' = \omega \cdot \cos \theta + l_u \cdot g({}_{rel}k_{\perp}, n_{\perp}) \quad , \quad (156)$$

$$\mp \frac{\bar{\omega}}{l_u} \cdot g(\tilde{n}'_{\perp}, m_{\perp}) = \mp \frac{\omega}{l_u} \cdot g(\tilde{n}_{\perp}, m_{\perp}) + g({}_{rel}k_{\perp}, m_{\perp}) \quad , \quad (157)$$

$$\mp \beta' \cdot \frac{\bar{\omega}}{l_u} = \mp \beta \cdot \frac{\omega}{l_u} + g({}_{rel}k_{\perp}, m_{\perp}) \quad , \quad (158)$$

$$\frac{\bar{\omega}}{l_u} \cdot \sin \theta' = \frac{\omega}{l_u} \cdot \sin \theta + g({}_{rel}k_{\perp}, m_{\perp}) \quad , \quad (159)$$

$$\bar{\omega} \cdot \sin \theta' = \omega \cdot \sin \theta + l_u \cdot g({}_{rel}k_{\perp}, m_{\perp}) \quad , \quad (160)$$

$$\omega = l_u \cdot g(\tilde{n}_{\perp}, k_{\perp}) \quad , \quad (161)$$

$$\begin{aligned} \frac{\bar{\omega}}{\omega} \cdot \cos \theta' &= \cos \theta + \frac{1}{\omega} \cdot l_u \cdot g({}_{rel}k_{\perp}, n_{\perp}) = \\ &= \cos \theta + \frac{g({}_{rel}k_{\perp}, n_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})} \quad , \end{aligned} \quad (162)$$

$$\begin{aligned} \frac{\bar{\omega}}{\omega} \cdot \sin \theta' &= \sin \theta + \frac{1}{\omega} \cdot l_u \cdot g({}_{rel}k_{\perp}, m_{\perp}) = \\ &= \sin \theta + \frac{g({}_{rel}k_{\perp}, m_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})} \quad , \end{aligned} \quad (163)$$

From the last (above) two relations, it follows for $tg\theta'$

$$tg\theta' = \frac{\sin \theta'}{\cos \theta'} = \frac{\sin \theta + \frac{g({}_{rel}k_{\perp}, m_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})}}{\cos \theta + \frac{g({}_{rel}k_{\perp}, n_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})}} \quad . \quad (164)$$

If we introduce the abbreviations

$$\bar{S} := \frac{g({}_{rel}k_{\perp}, m_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})} \quad , \quad (165)$$

$$\bar{C} := \frac{g({}_{rel}k_{\perp}, n_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})} \quad , \quad (166)$$

the expression for $tg\theta'$ could be written in the form

$$tg\theta' = \frac{\sin \theta'}{\cos \theta'} = \frac{\sin \theta + \bar{S}}{\cos \theta + \bar{C}} \quad . \quad (167)$$

The relations for $\sin \theta'$ and $\cos \theta'$ will have the forms respectively:

$$\frac{\bar{\omega}}{\omega} \cdot \sin \theta' = \sin \theta + \bar{S} \quad , \quad (168)$$

$$\frac{\bar{\omega}}{\omega} \cdot \cos \theta' = \cos \theta + \bar{C} \quad . \quad (169)$$

The angle θ' describes the deviation of the direction of the vector \bar{k}_\perp with respect to the vector \hat{k}_\perp . This type of deviation is usually related to the *aberration* of the wave vector \hat{k} during its motion in a time interval. As we will see below the aberration is depending on the relative velocity and relative acceleration included implicitly in the terms \bar{S} and \bar{C} .

From the expressions for $\sin \theta'$ and $\cos \theta'$ the relation between the emitted frequency $\bar{\omega}$ and the detected frequency ω follows in the forms

$$\frac{\bar{\omega}^2}{\omega^2} \cdot (\sin^2 \theta' + \cos^2 \theta') = \frac{\bar{\omega}^2}{\omega^2} = (\sin \theta + \bar{S})^2 + (\cos \theta + \bar{C})^2 \quad , \quad (170)$$

$$\bar{\omega} = [(\sin \theta + \bar{S})^2 + (\cos \theta + \bar{C})^2]^{1/2} \cdot \omega \quad . \quad (171)$$

The above relation describe the change of the frequency of the emitter during the motion of the signal from the emitter to the receiver (detector, observer) and is related to the description of the Doppler effect and the Hubble effect in spaces with affine connections and metrics considered as models of space-time. It is assumed that the emitter is at rest at a given time with respect to the frame of reference of an observer moving in space-time and receiving signals from an emitter. If at rest with respect to an observer, the emitter sends a signal with frequency $\bar{\omega}$ detected after a time interval as a frequency ω measured by the observer in his proper frame of reference.

The task is now to find out the explicit form for ${}_{rel}k_\perp$. For this purpose we would consider the change of a vector field ξ_\perp , orthogonal to the vector field u , i.e. $g(u, \xi_\perp) = 0$, transported along the vector field u .

4.1 Change of a non-isotropic vector field ξ_\perp along a non-isotropic vector field u , when $g(u, \xi_\perp) = 0$

Let us now consider the change of the vector field ξ_\perp , $g(u, \xi_\perp) = 0$, along the world line $x^i(\tau, \lambda_0)$ of an observer with tangent vector u . The vector ξ_\perp could be expressed at a point $A(\tau_0 - d\tau, \lambda_0)$ by means of the vector field ξ_\perp at the point $A'(\tau_0, \lambda_0)$ by the use of the covariant exponential operator ([3] - pp. 82-85) up to the second order of $d\tau$

$$\begin{aligned} \xi_{\perp(A)} & : = \xi_{\perp(\tau_0 - d\tau, \lambda_0)} := \bar{\xi}_{(\tau_0, \lambda_0)} = \\ & = \xi_{\perp(\tau_0, \lambda_0)} - d\tau \cdot \nabla_u \xi_{\perp}|_{(\tau_0, \lambda_0)} + \frac{1}{2} \cdot d\tau^2 \cdot \nabla_u \nabla_u \xi_{\perp}|_{(\tau_0, \lambda_0)} \quad .(172) \end{aligned}$$

The vector $\bar{\xi}_{\perp(\tau_0, \lambda_0)}$ could not be, in general, collinear to $\xi_{\perp(\tau_0, \lambda_0)}$ and orthogonal to $u_{(\tau_0, \lambda_0)}$. If we, further, consider the part of $\bar{\xi}_{(\tau_0, \lambda_0)}$, orthogonal to $u_{(\tau_0, \lambda_0)}$ at the point $A'(\tau_0, \lambda_0)$, we obtain

$$\begin{aligned} \bar{\xi}_{\perp(\tau_0, \lambda_0)} & : = \bar{g}[h_u(\bar{\xi}_{(\tau_0, \lambda_0)})] = \bar{g}[h_u(\xi_\perp)]_{(\tau_0, \lambda_0)} - \\ & - d\tau \cdot \bar{g}[h_u(\nabla_u \xi_\perp)]_{(\tau_0, \lambda_0)} + \frac{1}{2} \cdot d\tau^2 \cdot \bar{g}[h_u(\nabla_u \nabla_u \xi_\perp)]_{(\tau_0, \lambda_0)} \quad (173) \end{aligned}$$

Since,

$$\bar{g}[h_u(\xi_\perp)] = \xi_\perp , \quad (174)$$

$$\bar{g}[h_u(\nabla_u \xi_\perp)] = \text{rel}v , \quad (175)$$

$$\bar{g}[h_u(\nabla_u \nabla_u \xi_\perp)] = \text{rel}a , \quad (176)$$

we have the relation

$$\bar{\xi}_\perp(\tau_0, \lambda_0) = \xi_\perp(\tau_0, \lambda_0) - d\tau \cdot \text{rel}v(\tau_0, \lambda_0) + \frac{1}{2} \cdot d\tau^2 \cdot \text{rel}a(\tau_0, \lambda_0) . \quad (177)$$

Therefore, the vector $\xi_\perp(\tau_0 - d\tau, \lambda_0)$ with $g(\xi_\perp, u)_{(\tau_0 - d\tau, \lambda_0)} = 0$ could be considered as the vector $\bar{\xi}_\perp(\tau_0, \lambda_0)$ with $g(\bar{\xi}_\perp, u)_{(\tau_0, \lambda_0)} = 0$ if transported from the point $A(\tau_0 - d\tau, \lambda_0) \equiv A_{(\tau_0 - d\tau, \lambda_0)}$ to the point $A'(\tau_0, \lambda_0) \equiv A_{(\tau_0, \lambda_0)}$ at the world line with parameter τ .

Remark. Analogous considerations could be made with the transport of the vector ξ_\perp from the point $A''(\tau_0 + d\tau, \lambda_0)$ to the point $A(\tau_0, \lambda_0)$:

$$\xi_\perp(\tau_0 + d\tau, \lambda_0) := \bar{\bar{\xi}}_\perp(\tau_0, \lambda_0) = \xi_\perp(\tau_0, \lambda_0) + d\tau \cdot \text{rel}v(\tau_0, \lambda_0) + \frac{1}{2} \cdot d\tau^2 \cdot \text{rel}a(\tau_0, \lambda_0) . \quad (178)$$

If we summarize the expressions for $\bar{\xi}_\perp(\tau_0, \lambda_0)$ and $\bar{\bar{\xi}}_\perp(\tau_0, \lambda_0)$ we can find a relation between the vectors $\bar{\xi}_\perp(\tau_0, \lambda_0)$, $\bar{\bar{\xi}}_\perp(\tau_0, \lambda_0)$, $\bar{\xi}_\perp(\tau_0, \lambda_0)$, and the relative acceleration $\text{rel}a(\tau_0, \lambda_0)$

$$\bar{\bar{\xi}}_\perp(\tau_0, \lambda_0) + \bar{\xi}_\perp(\tau_0, \lambda_0) = 2 \cdot \xi_\perp(\tau_0, \lambda_0) + d\tau^2 \cdot \text{rel}a(\tau_0, \lambda_0) , \quad (179)$$

or

$$d\tau^2 \cdot \text{rel}a(\tau_0, \lambda_0) = \bar{\bar{\xi}}_\perp(\tau_0, \lambda_0) + \bar{\xi}_\perp(\tau_0, \lambda_0) - 2 \cdot \xi_\perp(\tau_0, \lambda_0) . \quad (180)$$

If the proper time interval $d\tau$ is expressed by the use of the relations between the proper time τ and the space distance dl , covered by a signal propagating with a velocity with absolute value l_u [from point of view of the observer with world line $x^i(\tau, \lambda_0)$], as

$$d\tau = \pm \frac{l_{\xi_\perp}}{l_u} \cdot d\lambda = \frac{dl}{l_u} \quad (181)$$

then the relation between $\bar{\xi}_\perp(\tau_0, \lambda_0)$ and $\xi_\perp(\tau_0, \lambda_0)$ could be written in the form

$$\bar{\xi}_\perp(\tau_0, \lambda_0) = \xi_\perp(\tau_0, \lambda_0) - \frac{dl}{l_u} \cdot \text{rel}v(\tau_0, \lambda_0) + \frac{1}{2} \cdot \left(\frac{dl}{l_u}\right)^2 \cdot \text{rel}a(\tau_0, \lambda_0) . \quad (182)$$

For every point $P'(\tau, \lambda_0)$ of the world line of the observer $x^i(\tau, \lambda_0)$ the above relation between $P'(\tau, \lambda_0)$ and another point $P(\tau - d\tau, \lambda_0)$ is valid. In the further consideration we will omit the indications for the corresponding points:

$$\bar{\xi}_\perp = \xi_\perp - \frac{dl}{l_u} \cdot \text{rel}v + \frac{1}{2} \cdot \left(\frac{dl}{l_u}\right)^2 \cdot \text{rel}a . \quad (183)$$

The vector $\bar{\xi}_\perp$ is, in general, not collinear to the vector ξ_\perp . It could be represented by the use of the unit vectors n_\perp and m_\perp in the form

$$\begin{aligned} \bar{\xi}_\perp &= l_{\bar{\xi}_\perp} \cdot \tilde{n}'_\perp = l_{\bar{\xi}_\perp} \cdot (\alpha' \cdot n_\perp + \beta' \cdot m_\perp) = \\ &= \mp l_{\bar{\xi}_\perp} \cdot [\cos \theta' \cdot n_\perp + \sin \theta' \cdot m_\perp] . \end{aligned} \quad (184)$$

On the other side, the vector ξ_\perp has the form $\xi_\perp = l_{\xi_\perp} \cdot n_\perp$, where $g(\xi_\perp, m_\perp) = l_{\xi_\perp} \cdot g(n_\perp, m_\perp) = 0$. In the same way, we can express the relative velocity $_{rel}v$ and the relative acceleration $_{rel}a$ by means of their structures, related to the vectors n_\perp and m_\perp .

If we project the expression for $\bar{\xi}_\perp$ along the unit vectors n_\perp and m_\perp correspondingly we obtain

$$\begin{aligned} g(\bar{\xi}_\perp, n_\perp) &= l_{\bar{\xi}_\perp} \cdot \cos \theta' = \\ &= g(\xi_\perp, n_\perp) - \frac{dl}{l_u} \cdot g(_{rel}v, n_\perp) + \frac{1}{2} \cdot \left(\frac{dl}{l_u}\right)^2 \cdot g(_{rel}a, n_\perp) = \\ &= \mp l_{\xi_\perp} - \frac{dl}{l_u} \cdot g(_{rel}v, n_\perp) + \frac{1}{2} \cdot \left(\frac{dl}{l_u}\right)^2 \cdot g(_{rel}a, n_\perp) , \end{aligned} \quad (185)$$

$$\begin{aligned} g(\bar{\xi}_\perp, m_\perp) &= l_{\bar{\xi}_\perp} \cdot \sin \theta' = \\ &= g(\xi_\perp, m_\perp) - \frac{dl}{l_u} \cdot g(_{rel}v, m_\perp) + \frac{1}{2} \cdot \left(\frac{dl}{l_u}\right)^2 \cdot g(_{rel}a, m_\perp) = \\ &= -\frac{dl}{l_u} \cdot g(_{rel}v, m_\perp) + \frac{1}{2} \cdot \left(\frac{dl}{l_u}\right)^2 \cdot g(_{rel}a, m_\perp) . \end{aligned} \quad (186)$$

We can find now the explicit forms of $g(_{rel}v, n_\perp)$, $g(_{rel}v, m_\perp)$, $g(_{rel}a, n_\perp)$, and $g(_{rel}a, m_\perp)$.

4.1.1 Explicit form of $l_{\bar{\xi}_\perp} \cdot \cos \theta'$ and $l_{\bar{\xi}_\perp} \cdot \sin \theta'$

Let us recall the explicit form of $_{rel}v$ and $_{rel}a$ with respect to their decompositions in centrifugal (centripetal) and Coriolis velocities and accelerations respectively [12].

The relative velocity $_{rel}v$ could be represented in the form

$$_{rel}v = v_z + v_c , \quad (187)$$

where

$$v_z = \mp l_{v_z} \cdot n_\perp = H \cdot l_{\xi_\perp} \cdot n_\perp = H \cdot \xi_\perp , \quad n_\perp = \frac{\xi_\perp}{l_{\xi_\perp}} , \quad (188)$$

$$v_c = \mp l_{v_c} \cdot m_\perp = H_c \cdot l_{\xi_\perp} \cdot m_\perp = H_c \cdot \eta_\perp , \quad m_\perp = \frac{\eta_\perp}{l_{\eta_\perp}} = \frac{v_c}{l_{v_c}} . \quad (189)$$

The relative acceleration $_{rel}a$ could be represented in the form

$$_{rel}a = a_z + a_c , \quad (190)$$

where

$$a_z = \mp l_{a_z} \cdot n_\perp = \bar{q} \cdot l_{\xi_\perp} \cdot n_\perp = \bar{q} \cdot \xi_\perp , \quad (191)$$

$$a_c = \mp l_{a_c} \cdot m_\perp = \bar{q}_c \cdot l_{\xi_\perp} \cdot m_\perp = \bar{q}_c \cdot \eta_\perp . \quad (192)$$

For the expressions $g(_{rel}v, n_\perp)$ and $g(_{rel}v, m_\perp)$ we obtain respectively

$$g(_{rel}v, n_\perp) = l_{v_z} = \mp H \cdot l_{\xi_\perp} , \quad (193)$$

$$g(_{rel}v, m_\perp) = l_{v_c} = \mp H_c \cdot l_{\xi_\perp} . \quad (194)$$

For the expressions $g(\text{rel}a, n_\perp)$ and $g(\text{rel}a, m_\perp)$ we obtain respectively

$$g(\text{rel}a, n_\perp) = l_{a_z} = \mp \bar{q} \cdot l_{\xi_\perp} \quad , \quad (195)$$

$$g(\text{rel}a, m_\perp) = l_{a_c} = \mp \bar{q}_c \cdot l_{\xi_\perp} \quad . \quad (196)$$

By means of the above relations, it follows for $l_{\xi_\perp}^- \cdot \cos \theta'$ and $l_{\xi_\perp}^- \cdot \sin \theta'$ respectively

$$\begin{aligned} l_{\xi_\perp}^- \cdot \cos \theta' &= \mp l_{\xi_\perp} - \frac{dl}{l_u} \cdot l_{v_z} + \frac{1}{2} \cdot \left(\frac{dl}{l_u}\right)^2 \cdot l_{a_z} = \\ &= \mp l_{\xi_\perp} \pm \frac{dl}{l_u} \cdot H \cdot l_{\xi_\perp} \mp \frac{1}{2} \cdot \left(\frac{dl}{l_u}\right)^2 \cdot \bar{q} \cdot l_{\xi_\perp} = \\ &= \mp l_{\xi_\perp} \cdot \left[1 - \frac{dl}{l_u} \cdot H + \frac{1}{2} \cdot \left(\frac{dl}{l_u}\right)^2 \cdot \bar{q}\right] \quad . \end{aligned} \quad (197)$$

$$\begin{aligned} l_{\xi_\perp}^- \cdot \sin \theta' &= -\frac{dl}{l_u} \cdot l_{v_c} + \frac{1}{2} \cdot \left(\frac{dl}{l_u}\right)^2 \cdot l_{a_c} = \\ &= \pm \frac{dl}{l_u} \cdot H_c \cdot l_{\xi_\perp} \mp \frac{1}{2} \cdot \left(\frac{dl}{l_u}\right)^2 \cdot \bar{q}_c \cdot l_{\xi_\perp} = \\ &= \pm l_{\xi_\perp} \cdot \left[\frac{dl}{l_u} \cdot H_c - \frac{1}{2} \cdot \left(\frac{dl}{l_u}\right)^2 \cdot \bar{q}_c\right] = \\ &= \mp l_{\xi_\perp} \cdot \left[-\frac{dl}{l_u} \cdot H_c + \frac{1}{2} \cdot \left(\frac{dl}{l_u}\right)^2 \cdot \bar{q}_c\right] \end{aligned} \quad (198)$$

If we introduce the abbreviations

$$C = -\frac{dl}{l_u} \cdot H + \frac{1}{2} \cdot \left(\frac{dl}{l_u}\right)^2 \cdot \bar{q} \quad , \quad (199)$$

$$S = -\frac{dl}{l_u} \cdot H_c + \frac{1}{2} \cdot \left(\frac{dl}{l_u}\right)^2 \cdot \bar{q}_c \quad , \quad (200)$$

the expressions for $l_{\xi_\perp}^- \cdot \sin \theta'$ and $l_{\xi_\perp}^- \cdot \cos \theta'$ could also be written in the forms

$$l_{\xi_\perp}^- \cdot \sin \theta' = \mp l_{\xi_\perp} \cdot S \quad , \quad (201)$$

$$l_{\xi_\perp}^- \cdot \cos \theta' = \mp l_{\xi_\perp} \cdot (1 + C) \quad . \quad (202)$$

The change of the direction of the vector ξ_\perp in the time interval $d\tau$ of the proper time τ of the observer on his world line can now be represented as

$$tg \theta' = \frac{S}{1 + C} \quad . \quad (203)$$

The change of the length of the vector ξ_\perp in the time interval $d\tau$ could be found in the form

$$l_{\xi_\perp}^2 = l_{\xi_\perp}^2 \cdot [(1 + C)^2 + S^2] \quad , \quad (204)$$

$$l_{\xi_\perp} = [(1 + C)^2 + S^2]^{1/2} \cdot l_{\xi_\perp} \quad , \quad (205)$$

$$l_{\xi_\perp} > 0 \quad , \quad l_{\xi_\perp} > 0 \quad . \quad (206)$$

Special case: The vector $\bar{\xi}_\perp$ is collinear to the vector ξ_\perp . Then $\tilde{n}'_\perp = n_\perp$, $\sin \theta' = 0$, $\cos \theta' = \mp 1$, $S = 0$, and

$$l_{\bar{\xi}_\perp} = (1 + C) \cdot l_{\xi_\perp} \quad . \quad (207)$$

Special case: The vector $\bar{\xi}_\perp$ is orthogonal to the vector ξ_\perp . Then $\tilde{n}'_\perp = m_\perp$, $\sin \theta' = \mp 1$, $\cos \theta' = 0$, $C = -1$, and

$$l_{\bar{\xi}_\perp} = S \cdot l_{\xi_\perp} \quad . \quad (208)$$

The change of the length of the vector field ξ_\perp along the world line of an observer shows the role of the relative velocity and the relative acceleration for the deformation of the vector field ξ_\perp along the world line. This deformation is depending on the corresponding Hubble functions H and H_c and acceleration parameters \bar{q} and \bar{q}_c .

Since the deformation of the contravariant non-isotropic vector field k_\perp (orthogonal to the vector field u) characterizes uniquely the deformation of the wave vector \tilde{k} we could consider the change of k_\perp along the world line of the observer in analogous way as it has been done for the vector field ξ_\perp .

4.2 Change of the vector k_\perp along the world line of an observer

Let us now consider the change of the vector field k_\perp , $g(u, k_\perp) = 0$, along the world line $x^i(\tau, \lambda_0)$ of an observer with tangent vector u . For this aim we will consider first of all the wave vector field \tilde{k} and then \tilde{k} will be projected to direction orthogonal to the vector field u .

The vector field \tilde{k} could be expressed at a point $A(\tau_0 - d\tau, \lambda_0)$ by means of the vector field \tilde{k} at the point $A'(\tau_0, \lambda_0)$ by the use of the covariant exponential operator ([3] - pp. 82-85) up to the second order of $d\tau$

$$\begin{aligned} \tilde{k}_{(A)} &: = \tilde{k}_{(\tau_0 - d\tau, \lambda_0)} := \bar{k}_{(\tau_0, \lambda_0)} = \\ &= \tilde{k}_{(\tau_0, \lambda_0)} - d\tau \cdot \nabla_u \tilde{k}|_{(\tau_0, \lambda_0)} + \frac{1}{2} \cdot d\tau^2 \cdot \nabla_u \nabla_u \tilde{k}|_{(\tau_0, \lambda_0)} \quad . \quad (209) \end{aligned}$$

The orthogonal to u parts of \tilde{k} and \bar{k} at the point $A'(\tau_0, \lambda_0)$ could be found after projection of both the vectors by means of the projection metric h_u

$$\bar{g}[h_u(\tilde{k})]_{(\tau_0 - d\tau, \lambda_0)} : = \bar{g}[h_u(\bar{k})]_{(\tau_0, \lambda_0)} = \quad (210)$$

$$\begin{aligned} &= \bar{g}[h_u(\tilde{k})]_{(\tau_0, \lambda_0)} - d\tau \cdot \bar{g}[h_u(\nabla_u \tilde{k})]_{|(\tau_0, \lambda_0)} + \\ &+ \frac{1}{2} \cdot d\tau^2 \cdot \bar{g}[h_u(\nabla_u \nabla_u \tilde{k})]_{|(\tau_0, \lambda_0)} \quad . \quad (211) \end{aligned}$$

Since,

$$\bar{g}[h_u(\bar{k})]_{(\tau_0, \lambda_0)} = \bar{k}_\perp(\tau_0, \lambda_0) \quad , \quad (212)$$

$$\bar{g}[h_u(\tilde{k})]_{(\tau_0, \lambda_0)} = k_\perp(\tau_0, \lambda_0) \quad , \quad (213)$$

$$\bar{g}[h_u(\nabla_u \tilde{k})]_{|(\tau_0, \lambda_0)} = (\nabla_u \tilde{k})_\perp|_{(\tau_0, \lambda_0)} \quad , \quad (214)$$

$$\bar{g}[h_u(\nabla_u \nabla_u \tilde{k})]_{|(\tau_0, \lambda_0)} = (\nabla_u \nabla_u \tilde{k})_\perp|_{(\tau_0, \lambda_0)} \quad , \quad (215)$$

$$\bar{\omega}_{(\tau_0, \lambda_0)} = g(\bar{k}, u)_{(\tau_0, \lambda_0)} \quad , \quad \omega_{(\tau_0, \lambda_0)} = g(\tilde{k}, u)_{(\tau_0, \lambda_0)} \quad , \quad (216)$$

we have the relation

$$\begin{aligned} \bar{k}_{\perp(\tau_0, \lambda_0)} &= k_{\perp(\tau_0, \lambda_0)} - d\tau \cdot \bar{g}[h_u(\nabla_u \tilde{k})]_{|(\tau_0, \lambda_0)} + \\ &+ \frac{1}{2} \cdot d\tau^2 \cdot \bar{g}[h_u(\nabla_u \nabla_u \tilde{k})]_{|(\tau_0, \lambda_0)} \quad . \end{aligned} \quad (217)$$

The vectors $\bar{g}[h_u(\nabla_u \tilde{k})]_{(\tau_0, \lambda_0)}$ and $\bar{g}[h_u(\nabla_u \nabla_u \tilde{k})]_{(\tau_0, \lambda_0)}$ could be represented by the use of the kinematic characteristics of the relative velocity and relative acceleration [3] in the forms

$$\begin{aligned} \bar{g}[h_u(\nabla_u \tilde{k})] &= \bar{g}[h_u(\frac{g(u, \tilde{k})}{e} \cdot a + \mathcal{L}_u \tilde{k}) + d(\tilde{k})] = \\ &= \bar{g}[h_u(\pm \frac{\omega}{l_u^2} \cdot a + \mathcal{L}_u \tilde{k}) + d(\tilde{k})] = \\ &= \bar{g}[\pm \frac{\omega}{l_u^2} \cdot h_u(a) + h_u(\mathcal{L}_u \tilde{k}) + d(\tilde{k})] \quad , \end{aligned} \quad (218)$$

$$\begin{aligned} \bar{g}[h_u(\nabla_u \nabla_u \tilde{k})] &= \bar{g}\{h_u[\frac{g(u, \tilde{k})}{e} \cdot \nabla_u a + k(g)(\mathcal{L}_u \tilde{k}) + \nabla_u(\mathcal{L}_u \tilde{k})] + A(\tilde{k})\} = \\ &= \bar{g}\{\pm \frac{\omega}{l_u^2} \cdot h_u(\nabla_u a) + h_u[k(g)(\mathcal{L}_u \tilde{k})] + h_u[\nabla_u(\mathcal{L}_u \tilde{k})] + \\ &+ A(\tilde{k})\} \quad . \end{aligned} \quad (219)$$

If we assume that $\mathcal{L}_u \tilde{k} = 0$ we obtain the relations

$$\begin{aligned} \bar{g}[h_u(\nabla_u \tilde{k})] &= \bar{g}[\pm \frac{\omega}{l_u^2} \cdot h_u(a) + d(\tilde{k})] = \\ &= \pm \frac{\omega}{l_u^2} \cdot \bar{g}[h_u(a)] + \bar{g}[d(k_{\perp})] = \\ &= \pm \frac{\omega}{l_u^2} \cdot a_{\perp} + \bar{g}[d(k_{\perp})] \quad , \end{aligned} \quad (220)$$

$$\begin{aligned} \bar{g}[h_u(\nabla_u \nabla_u \tilde{k})] &= \pm \frac{\omega}{l_u^2} \cdot \bar{g}[h_u(\nabla_u a)] + \bar{g}[A(k_{\perp})] = \\ &= \pm \frac{\omega}{l_u^2} \cdot (\nabla_u a)_{\perp} + \bar{g}[A(k_{\perp})] \quad , \end{aligned} \quad (221)$$

where

$$\frac{g(u, \tilde{k})}{e} = \frac{\omega}{\pm l_u^2} = \pm \frac{\omega}{l_u^2} \quad , \quad (222)$$

$$a_{\perp} = \bar{g}[h_u(a)] \quad , \quad (223)$$

$$(\nabla_u a)_{\perp} = \bar{g}[h_u(\nabla_u a)] \quad , \quad (224)$$

$$\bar{g}[d(\tilde{k})] = \bar{g}[d(k_{\perp})] \quad , \quad (225)$$

$$\bar{g}[A(\tilde{k})] = \bar{g}[A(k_{\perp})] \quad . \quad (226)$$

Remark. The condition $\mathcal{L}_u \tilde{k} = 0$ assures the possibility for introducing coordinates with tangent vectors u and \tilde{k} respectively.

On the other side, the vector k_\perp could be represented in its projections along the vectors n_\perp and m_\perp in the form

$$k_\perp = \mp l_{k_\perp} \cdot \tilde{n}_\perp = \mp \frac{\omega}{l_u} \cdot \tilde{n}_\perp , \quad (227)$$

$$\tilde{n}_\perp = \alpha \cdot n_\perp + \beta \cdot m_\perp = \mp [\cos \theta \cdot n_\perp + \sin \theta \cdot m_\perp] . \quad (228)$$

Then

$$k_\perp = \frac{\omega}{l_u} \cdot [\cos \theta \cdot n_\perp + \sin \theta \cdot m_\perp] . \quad (229)$$

In analogous way, the vector \bar{k}_\perp could be represented in the form

$$\bar{k}_\perp = \mp l_{\bar{k}_\perp} \cdot \tilde{n}'_\perp = \mp \frac{\bar{\omega}}{l_u} \cdot \tilde{n}'_\perp , \quad (230)$$

$$\tilde{n}'_\perp = \alpha' \cdot n_\perp + \beta' \cdot m_\perp = \mp [\cos \theta' \cdot n_\perp + \sin \theta' \cdot m_\perp] , \quad (231)$$

$$\bar{k}_\perp = \frac{\bar{\omega}}{l_u} \cdot [\cos \theta' \cdot n_\perp + \sin \theta' \cdot m_\perp] . \quad (232)$$

The representation of \bar{k}_\perp in the form $\bar{k}_\perp = k_\perp + {}_{rel}k_\perp$ will be given now in the form

$$\bar{k}_\perp = k_\perp + {}_{rel}k_\perp = \quad (233)$$

$$= k_\perp - d\tau \cdot \bar{g}[h_u(\nabla_u \tilde{k})] + \frac{1}{2} \cdot d\tau^2 \cdot \bar{g}[h_u(\nabla_u \nabla_u \tilde{k})] , \quad (234)$$

where

$$\begin{aligned} {}_{rel}k_\perp &= -d\tau \cdot (\nabla_u \tilde{k})_\perp + \frac{1}{2} \cdot d\tau^2 \cdot (\nabla_u \nabla_u \tilde{k})_\perp = \\ &= -d\tau \cdot \left\{ \pm \frac{\omega}{l_u^2} \cdot a_\perp + \bar{g}[d(k_\perp)] \right\} + \\ &+ \frac{1}{2} \cdot d\tau^2 \cdot \left\{ \pm \frac{\omega}{l_u^2} \cdot (\nabla_u a)_\perp + \bar{g}[A(k_\perp)] \right\} , \end{aligned} \quad (235)$$

$$(\nabla_u \tilde{k})_\perp = \bar{g}[h_u(\nabla_u \tilde{k})] , \quad (236)$$

$$(\nabla_u \nabla_u \tilde{k})_\perp = \bar{g}[h_u(\nabla_u \nabla_u \tilde{k})] . \quad (237)$$

The terms in ${}_{rel}k_\perp$ could be further represented by means of the structures of the relative velocity and the relative acceleration corresponding to the centrifugal (centripetal) and Coriolis velocities and accelerations.

4.2.1 Representation of ${}_{rel}k_\perp$ by means of the centrifugal (centripetal) and Coriolis velocities and accelerations

1. The orthogonal to u acceleration a_\perp could be found after projection by the use of the projective metrics

$$h_{\xi_\perp} = g - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot g(\xi_\perp) \otimes g(\xi_\perp) , \quad (238)$$

$$h^{\xi_\perp} = \bar{g} - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp \otimes \xi_\perp , \quad (239)$$

in parts collinear to the vector field ξ_\perp and orthogonal to ξ_\perp . At the same time both the parts are orthogonal to the vector field u .

$$a_\perp = \frac{g(a_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + \bar{g}[h_{\xi_\perp}(a_\perp)] = \quad (240)$$

$$\begin{aligned} &= \frac{l_{\xi_\perp}^2}{\mp l_{\xi_\perp}^2} \cdot g(a_\perp, n_\perp) \cdot n_\perp + \bar{g}[h_{\xi_\perp}(a_\perp)] = \\ &= \mp g(a_\perp, n_\perp) \cdot n_\perp + \bar{g}[h_{\xi_\perp}(a_\perp)] = \\ &= (a_\perp)_z + (a_\perp)_c, \end{aligned} \quad (241)$$

where

$$(a_\perp)_z = \mp g(a_\perp, n_\perp) \cdot n_\perp, \quad (242)$$

$$(a_\perp)_c = \bar{g}[h_{\xi_\perp}(a_\perp)], \quad (243)$$

$$g(\xi_\perp, (a_\perp)_c) = 0, \quad (244)$$

$$(a_\perp)_z = \mp l_{(a_\perp)_z} \cdot n_\perp, \quad (245)$$

$$(a_\perp)_c = \mp l_{(a_\perp)_c} \cdot m_\perp. \quad (246)$$

2. The orthogonal to u change $(\nabla_u a)_\perp$ of the acceleration a along u could be found after projection by the use of the projective metrics h_{ξ_\perp} and h^{ξ_\perp} in parts collinear to the vector field ξ_\perp and orthogonal to ξ_\perp . At the same time both the parts are orthogonal to the vector field u .

$$\begin{aligned} (\nabla_u a)_\perp &= \frac{g((\nabla_u a)_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + \bar{g}[h_{\xi_\perp}(\nabla_u a)_\perp] = \\ &= \mp g((\nabla_u a)_\perp, n_\perp) \cdot n_\perp + \bar{g}[h_{\xi_\perp}(\nabla_u a)_\perp] = \\ &= (\nabla_u a)_{\perp z} + (\nabla_u a)_{\perp c}, \end{aligned} \quad (247)$$

where

$$(\nabla_u a)_{\perp z} = \mp g((\nabla_u a)_\perp, n_\perp) \cdot n_\perp, \quad (248)$$

$$(\nabla_u a)_{\perp c} = \bar{g}[h_{\xi_\perp}(\nabla_u a)_\perp], \quad (249)$$

$$g(\xi_\perp, (\nabla_u a)_{\perp c}) = 0, \quad (250)$$

$$(\nabla_u a)_{\perp z} = \mp l_{(\nabla_u a)_{\perp z}} \cdot n_\perp, \quad (251)$$

$$(\nabla_u a)_{\perp c} = \mp l_{(\nabla_u a)_{\perp c}} \cdot m_\perp. \quad (252)$$

3. The orthogonal to u deformation velocity vector $\bar{g}[d(k_\perp)]$ could be found after projection by the use of the projective metrics h_{ξ_\perp} and h^{ξ_\perp} in parts collinear to the vector field ξ_\perp and orthogonal to ξ_\perp . At the same time both the parts are orthogonal to the vector field u . Since $k_\perp = \mp l_{k_\perp} \cdot \tilde{n}_\perp$ we have the relations

$$\begin{aligned} d(k_\perp) &= d(\mp l_{k_\perp} \cdot \tilde{n}_\perp) = \mp l_{k_\perp} \cdot d(\tilde{n}_\perp) = \\ &= \mp l_{k_\perp} \cdot d(\alpha \cdot n_\perp + \beta \cdot m_\perp) = \\ &= \mp l_{k_\perp} \cdot [\alpha \cdot d(n_\perp) + \beta \cdot d(m_\perp)], \end{aligned} \quad (253)$$

$$\bar{g}[d(n_\perp)] = \frac{g(\bar{g}[d(n_\perp)], \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + \bar{g}[h_{\xi_\perp}(\bar{g}[d(n_\perp)])], \quad (254)$$

$$\begin{aligned}
\bar{g}[d(n_\perp)] &= \frac{g(\bar{g}[d(n_\perp)], n_\perp)}{\mp l_{\xi_\perp}^2} \cdot l_{\xi_\perp}^2 \cdot n_\perp + \bar{g}[h_{\xi_\perp}(\bar{g}[d(n_\perp)])] = \\
&= \mp g(\bar{g}[d(n_\perp)], n_\perp) \cdot n_\perp + \bar{g}[h_{\xi_\perp}(\bar{g}[d(n_\perp)])] , \quad (255)
\end{aligned}$$

$$g(\bar{g}[d(n_\perp)], n_\perp) = g_{ij} \cdot g^{ik} \cdot d_{kl} \cdot n_\perp^l \cdot n_\perp^j = d_{jl} \cdot n_\perp^j \cdot n_\perp^l = d(n_\perp, n_\perp) , \quad (256)$$

$$\mp g(\bar{g}[d(n_\perp)], n_\perp) \cdot n_\perp = \mp d(n_\perp, n_\perp) \cdot n_\perp . \quad (257)$$

On the other side, the following relations can be proved:

$$\begin{aligned}
h^{\xi_\perp} &= \bar{g} - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp \otimes \xi_\perp = \\
&= \bar{g} - \frac{1}{\mp l_{\xi_\perp}^2} \cdot l_{\xi_\perp} \cdot l_{\xi_\perp} \cdot n_\perp \otimes n_\perp = \\
&= \bar{g} \pm n_\perp \otimes n_\perp = \bar{g} - \frac{1}{g(n_\perp, n_\perp)} \cdot n_\perp \otimes n_\perp = \\
&= h^{n_\perp} , \quad (258)
\end{aligned}$$

$$\begin{aligned}
h^u &= \bar{g} - \frac{1}{g(u, u)} \cdot u \otimes u = \\
&= \bar{g} - \frac{1}{\pm l_u^2} \cdot l_u \cdot l_u \cdot n_\parallel \otimes n_\parallel = \\
&= \bar{g} \mp n_\parallel \otimes n_\parallel = \bar{g} - \frac{1}{g(n_\parallel, n_\parallel)} \cdot n_\parallel \otimes n_\parallel = \\
&= h^{n_\parallel} , \quad (259)
\end{aligned}$$

$$\begin{aligned}
h_{\xi_\perp} &= g - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot g(\xi_\perp) \otimes g(\xi_\perp) = \\
&= g - \frac{1}{\mp l_{\xi_\perp}^2} \cdot l_{\xi_\perp} \cdot l_{\xi_\perp} \cdot g(n_\perp) \otimes (n_\perp) = \\
&= g - \frac{1}{g(n_\perp, n_\perp)} \cdot g(n_\perp) \otimes (n_\perp) = \\
&= h_{n_\perp} , \quad (260)
\end{aligned}$$

$$\begin{aligned}
h_u &= g - \frac{1}{g(u, u)} \cdot g(u) \otimes g(u) = \\
&= g - \frac{1}{\pm l_u^2} \cdot l_u \cdot l_u \cdot g(n_\parallel) \otimes g(n_\parallel) = \\
&= g - \frac{1}{g(n_\parallel, n_\parallel)} \cdot g(n_\parallel) \otimes g(n_\parallel) = \\
&= h_{n_\parallel} . \quad (261)
\end{aligned}$$

By the use of the above expressions we can find the explicit form of the term $\overline{g}[h_{\xi_{\perp}}(\overline{g}[d(n_{\perp})])]$:

$$\begin{aligned}
\overline{g}[h_{\xi_{\perp}}(\overline{g}[d(n_{\perp})])] &= g^{ij} \cdot (h_{\xi_{\perp}})_{\overline{jk}} \cdot g^{kl} \cdot d_{\overline{lm}} \cdot n_{\perp}^m \cdot \partial_i = \\
&= g^{ij} \cdot (g_{\overline{jk}} - \frac{1}{g(\xi_{\perp}, \xi_{\perp})} \cdot g_{\overline{jl}} \cdot \xi_{\perp}^l \cdot g_{\overline{kr}} \cdot \xi_{\perp}^r) \cdot g^{kl} \cdot d_{\overline{lm}} \cdot n_{\perp}^m \cdot \partial_i = \\
&= (g_k^i - \frac{1}{g(\xi_{\perp}, \xi_{\perp})} \cdot g_l^i \cdot \xi_{\perp}^l \cdot g_{\overline{kr}} \cdot \xi_{\perp}^r) \cdot g^{kl} \cdot d_{\overline{lm}} \cdot n_{\perp}^m \cdot \partial_i = \\
&= (g^{il} - \frac{1}{g(\xi_{\perp}, \xi_{\perp})} \cdot \xi_{\perp}^i \cdot g_r^l \cdot \xi_{\perp}^r) \cdot d_{\overline{lm}} \cdot n_{\perp}^m \cdot \partial_i = \\
&= (g^{il} - \frac{1}{g(\xi_{\perp}, \xi_{\perp})} \cdot \xi_{\perp}^i \cdot \xi_{\perp}^l) \cdot d_{\overline{lm}} \cdot n_{\perp}^m \cdot \partial_i = \\
&= h^{\xi_{\perp}}[d(n_{\perp})] = h^{n_{\perp}}[d(n_{\perp})] \quad . \tag{262}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{g}[d(n_{\perp})] &= \mp g(\overline{g}[d(n_{\perp})], n_{\perp}) \cdot n_{\perp} + \overline{g}[h_{\xi_{\perp}}(\overline{g}[d(n_{\perp})])] = \\
&= \mp d(n_{\perp}, n_{\perp}) \cdot n_{\perp} + h^{n_{\perp}}[d(n_{\perp})] \quad . \tag{263}
\end{aligned}$$

On the other side, the term $\overline{g}[d(m_{\perp})]$ could be projected in an analogous way

$$\overline{g}[d(m_{\perp})] = \mp g(\overline{g}[d(m_{\perp})], n_{\perp}) \cdot n_{\perp} + \overline{g}[h_{\xi_{\perp}}(\overline{g}[d(m_{\perp})])] \quad . \tag{264}$$

The terms at the right side of the expression could be found in the corresponding forms by the use of the relations

$$g(\overline{g}[d(m_{\perp})], n_{\perp}) = g_{\overline{ij}} \cdot g^{ik} \cdot d_{\overline{kl}} \cdot m_{\perp}^l \cdot n_{\perp}^j = \tag{265}$$

$$= d_{\overline{jl}} \cdot m_{\perp}^l \cdot n_{\perp}^j = d(n_{\perp}, m_{\perp}) \quad ,$$

$$\mp g(\overline{g}[d(n_{\perp})], n_{\perp}) \cdot n_{\perp} = \mp d(n_{\perp}, m_{\perp}) \cdot n_{\perp} \quad , \tag{266}$$

$$\overline{g}[h_{\xi_{\perp}}(\overline{g}[d(m_{\perp})])] = g^{ij} \cdot (h_{\xi_{\perp}})_{\overline{jk}} \cdot g^{kl} \cdot d_{\overline{lr}} \cdot m_{\perp}^r \cdot \partial_i = \tag{267}$$

$$= \overline{g}(h_{\xi_{\perp}})(\overline{g})[d(m_{\perp})] \quad , \tag{268}$$

$$\overline{g}(h_{\xi_{\perp}})(\overline{g}) = h^{\xi_{\perp}} = h^{n_{\perp}} \quad , \tag{269}$$

$$\overline{g}[h_{\xi_{\perp}}(\overline{g}[d(m_{\perp})])] = h^{n_{\perp}}[d(m_{\perp})] \quad . \tag{270}$$

Therefore,

$$\overline{g}[d(m_{\perp})] = \mp d(n_{\perp}, m_{\perp}) \cdot n_{\perp} + h^{n_{\perp}}[d(m_{\perp})] \quad . \tag{271}$$

For $\overline{g}[d(\tilde{n}_{\perp})]$ the expressions follow

$$\begin{aligned}
\overline{g}[d(\tilde{n}_{\perp})] &= \alpha \cdot \overline{g}[d(n_{\perp})] + \beta \cdot \overline{g}[d(m_{\perp})] = \\
&= \mp \alpha \cdot d(n_{\perp}, n_{\perp}) \cdot n_{\perp} + \alpha \cdot h^{n_{\perp}}[d(n_{\perp})] \mp \\
&\quad \mp \beta \cdot d(n_{\perp}, m_{\perp}) \cdot n_{\perp} + \beta \cdot h^{n_{\perp}}[d(m_{\perp})] \tag{272}
\end{aligned}$$

$$\begin{aligned}
\overline{g}[d(\tilde{n}_{\perp})] &= \mp [\alpha \cdot d(n_{\perp}, n_{\perp}) + \beta \cdot d(n_{\perp}, m_{\perp})] \cdot n_{\perp} + \\
&\quad + \alpha \cdot h^{n_{\perp}}[d(n_{\perp})] + \beta \cdot h^{n_{\perp}}[d(m_{\perp})] \quad . \tag{273}
\end{aligned}$$

On the other side, the structures of ${}_{rel}v = v_z + v_c$ could be represented under the condition $\mathcal{L}_u \xi_\perp = 0$ in the forms

$${}_{rel}v = l_{\xi_\perp} \cdot \bar{g}[d(n_\perp)] \quad , \quad (274)$$

$$v_z = \mp l_{\xi_\perp} \cdot d(n_\perp, n_\perp) \cdot n_\perp \quad , \quad (275)$$

$$\begin{aligned} v_c &= \bar{g}[h_{\xi_\perp}({}_{rel}v)] = l_{\xi_\perp} \cdot \bar{g}(h_{\xi_\perp})(\bar{g})[d(n_\perp)] = l_{\xi_\perp} \cdot h^{n_\perp}[d(n_\perp)] \\ g(v_c, n_\perp) &= 0 \quad , \end{aligned} \quad (276)$$

$$\begin{aligned} {}_{rel}v &= v_z + v_c = l_{\xi_\perp} \cdot \bar{g}[d(n_\perp)] = \\ &= \mp l_{\xi_\perp} \cdot d(n_\perp, n_\perp) \cdot n_\perp + l_{\xi_\perp} \cdot h^{n_\perp}[d(n_\perp)] \quad . \end{aligned} \quad (277)$$

Let us introduce now a vector field $\eta_\perp = l_{\xi_\perp} \cdot m_\perp$, orthogonal to the vector field $\xi_\perp = l_{\xi_\perp} \cdot n_\perp$ and u , but with the same length as ξ_\perp , i.e.

$$g(\eta_\perp, \xi_\perp) = l_{\xi_\perp}^2 \cdot g(m_\perp, n_\perp) = 0 \quad , \quad g(\eta_\perp, \eta_\perp) = \mp l_{\xi_\perp}^2 \quad . \quad (278)$$

The corresponding to η_\perp relative velocity ${}_{rel}v_\eta$ and relative acceleration ${}_{rel}a_\eta$ have analogous forms as ${}_{rel}v$ and ${}_{rel}a$.

$${}_{rel}v_\eta = \bar{g}[d(\eta_\perp)] = l_{\xi_\perp} \cdot \bar{g}[d(m_\perp)] \quad , \quad (279)$$

$${}_{rel}a_\eta = \bar{g}[A(\eta_\perp)] = l_{\xi_\perp} \cdot \bar{g}[A(m_\perp)] \quad . \quad (280)$$

The decomposition of ${}_{rel}v_\eta$ has the form

$${}_{rel}v_\eta = v_{\eta z} + v_{\eta c} \quad , \quad (281)$$

$$\begin{aligned} {}_{rel}v_\eta &= \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot g({}_{rel}v_\eta, \xi_\perp) \cdot \xi_\perp + \bar{g}[h_{\xi_\perp}({}_{rel}v_\eta)] = v_{\eta z} + v_{\eta c} = \\ &= \mp g({}_{rel}v_\eta, n_\perp) \cdot n_\perp + \bar{g}[h_{\xi_\perp}({}_{rel}v_\eta)] \end{aligned} \quad (282)$$

where

$$v_{\eta z} = \mp g({}_{rel}v_\eta, n_\perp) \cdot n_\perp \quad , \quad (283)$$

$$v_{\eta c} = \bar{g}[h_{\xi_\perp}({}_{rel}v_\eta)] \quad . \quad (284)$$

The explicit form of $v_{\eta z}$ and $v_{\eta c}$ could be found by the use of the relations under the condition $\mathcal{L}_u \eta_\perp = 0$

$$\begin{aligned} g({}_{rel}v_\eta, n_\perp) &= g(\bar{g}[d(\eta_\perp)], n_\perp) = g_{ij}^{-1} \cdot g^{ik} \cdot d_{kl}^{-1} \cdot l_{\xi_\perp}^l \cdot m_\perp^l \cdot n_\perp^j = \\ &= l_{\xi_\perp} \cdot d_{jl}^{-1} \cdot n_\perp^j \cdot m_\perp^l = l_{\xi_\perp} \cdot d(n_\perp, m_\perp) \quad , \end{aligned} \quad (285)$$

$$\bar{g}[h_{\xi_\perp}({}_{rel}v_\eta)] = l_{\xi_\perp} \cdot \bar{g}[h_{\xi_\perp}(\bar{g}[d(m_\perp)])] = l_{\xi_\perp} \cdot h^{n_\perp}[d(m_\perp)] \quad , \quad (286)$$

as

$$v_{\eta z} = \mp g({}_{rel}v_\eta, n_\perp) \cdot n_\perp = \mp l_{\xi_\perp} \cdot d(n_\perp, m_\perp) \cdot n_\perp \quad , \quad (287)$$

$$v_{\eta c} = \bar{g}[h_{\xi_\perp}({}_{rel}v_\eta)] = l_{\xi_\perp} \cdot h^{n_\perp}[d(m_\perp)] \quad . \quad (288)$$

Now we can find the relations between the relative velocities ${}_{rel}v$, ${}_{rel}v_\eta$, and the expression for $\bar{g}[d(\tilde{n}_\perp)]$

$$\begin{aligned}\bar{g}[d(\tilde{n}_\perp)] &= \alpha \cdot \bar{g}[d(n_\perp)] + \beta \cdot \bar{g}[d(m_\perp)] = \\ &= \mp \alpha \cdot d(n_\perp, n_\perp) \cdot n_\perp + \alpha \cdot h^{n_\perp}[d(n_\perp)] \mp \\ &\quad \mp \beta \cdot d(n_\perp, m_\perp) \cdot n_\perp + \beta \cdot h^{n_\perp}[d(m_\perp)] \quad ,\end{aligned}\quad (289)$$

$$\begin{aligned}\bar{g}[d(\tilde{n}_\perp)] &= \mp \alpha \cdot (\mp \frac{1}{l_{\xi_\perp}} \cdot v_z) + \alpha \cdot \frac{1}{l_{\xi_\perp}} \cdot v_c \mp \\ &\quad \mp \beta \cdot (\mp \frac{1}{l_{\xi_\perp}} \cdot v_{\eta z}) + \beta \cdot \frac{1}{l_{\xi_\perp}} \cdot v_{\eta c} \quad ,\end{aligned}\quad (290)$$

$$\begin{aligned}\bar{g}[d(\tilde{n}_\perp)] &= \alpha \cdot \frac{1}{l_{\xi_\perp}} \cdot v_z + \alpha \cdot \frac{1}{l_{\xi_\perp}} \cdot v_c + \\ &\quad + \beta \cdot \frac{1}{l_{\xi_\perp}} \cdot v_{\eta z} + \beta \cdot \frac{1}{l_{\xi_\perp}} \cdot v_{\eta c} \quad ,\end{aligned}\quad (291)$$

$$\begin{aligned}\bar{g}[d(\tilde{n}_\perp)] &= \frac{1}{l_{\xi_\perp}} \cdot [\alpha \cdot (v_z + v_c) + \beta \cdot (v_{\eta z} + v_{\eta c})] = \\ &= \frac{1}{l_{\xi_\perp}} \cdot (\alpha \cdot {}_{rel}v + \beta \cdot {}_{rel}v_\eta) \quad .\end{aligned}\quad (292)$$

4. The orthogonal to u deformation acceleration vector $\bar{g}[A(k_\perp)]$ could be found after projection by the use of the projective metrics h_{ξ_\perp} and h^{ξ_\perp} in parts collinear to the vector field ξ_\perp and orthogonal to ξ_\perp . At the same time both the parts are orthogonal to the vector field u . Since $k_\perp = \mp l_{k_\perp} \cdot \tilde{n}_\perp$ we have the relations

$$\begin{aligned}A(k_\perp) &= A(\mp l_{k_\perp} \cdot \tilde{n}_\perp) = \mp l_{k_\perp} \cdot A(\tilde{n}_\perp) = \\ &= \mp l_{k_\perp} \cdot A(\alpha \cdot n_\perp + \beta \cdot m_\perp) = \\ &= \mp l_{k_\perp} \cdot [\alpha \cdot A(n_\perp) + \beta \cdot A(m_\perp)] \quad ,\end{aligned}\quad (293)$$

$$\bar{g}[A(n_\perp)] = \frac{g(\bar{g}[A(n_\perp)], \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + \bar{g}[h_{\xi_\perp}(\bar{g}[A(n_\perp)])] \quad ,\quad (294)$$

$$\begin{aligned}\bar{g}[A(n_\perp)] &= \frac{g(\bar{g}[A(n_\perp)], n_\perp)}{\mp l_{\xi_\perp}^2} \cdot l_{\xi_\perp}^2 \cdot n_\perp + \bar{g}[h_{\xi_\perp}(\bar{g}[A(n_\perp)])] = \\ &= \mp g(\bar{g}[A(n_\perp)], n_\perp) \cdot n_\perp + \bar{g}[h_{\xi_\perp}(\bar{g}[A(n_\perp)])] \quad ,\end{aligned}\quad (295)$$

$$g(\bar{g}[A(n_\perp)], n_\perp) = g_{ij} \cdot g^{ik} \cdot A_{kl} \cdot n_\perp^l \cdot n_\perp^j = A_{jl} \cdot n_\perp^j \cdot n_\perp^l = A(n_\perp, n_\perp) \quad ,\quad (296)$$

$$\mp g(\bar{g}[A(n_\perp)], n_\perp) \cdot n_\perp = \mp A(n_\perp, n_\perp) \cdot n_\perp \quad .\quad (297)$$

Therefore,

$$\begin{aligned}\bar{g}[A(n_\perp)] &= \mp g(\bar{g}[A(n_\perp)], n_\perp) \cdot n_\perp + \bar{g}[h_{\xi_\perp}(\bar{g}[A(n_\perp)])] = \\ &= \mp A(n_\perp, n_\perp) \cdot n_\perp + h^{n_\perp}[A(n_\perp)] \quad .\end{aligned}\quad (298)$$

On the other side, the term $\bar{g}[A(m_\perp)]$ could be projected in an analogous way

$$\bar{g}[A(m_\perp)] = \mp g(\bar{g}[A(m_\perp)], n_\perp) \cdot n_\perp + \bar{g}[h_{\xi_\perp}(\bar{g}[A(m_\perp)])] \quad . \quad (299)$$

The terms at the left side of the expression could be found in the corresponding forms by the use of the relations

$$\begin{aligned}g(\bar{g}[A(m_\perp)], n_\perp) &= g_{ij} \cdot g^{ik} \cdot A_{kl} \cdot m_\perp^l \cdot n_\perp^j = \\ &= A_{jl} \cdot m_\perp^l \cdot n_\perp^j = A(n_\perp, m_\perp) \quad ,\end{aligned}\quad (300)$$

$$\mp g(\bar{g}[A(n_\perp)], n_\perp) \cdot n_\perp = \mp A(n_\perp, m_\perp) \cdot n_\perp \quad , \quad (301)$$

$$\bar{g}[h_{\xi_\perp}(\bar{g}[A(m_\perp)])] = g^{ij} \cdot (h_{\xi_\perp})_{jk} \cdot g^{kl} \cdot A_{lr} \cdot m_\perp^r \cdot \partial_i = \quad (302)$$

$$= \bar{g}(h_{\xi_\perp})(\bar{g})[A(m_\perp)] \quad , \quad (303)$$

$$\bar{g}(h_{\xi_\perp})(\bar{g}) = h^{\xi_\perp} = h^{n_\perp} \quad , \quad (304)$$

$$\bar{g}[h_{\xi_\perp}(\bar{g}[A(m_\perp)])] = h^{n_\perp}[A(m_\perp)] \quad . \quad (305)$$

Therefore,

$$\bar{g}[A(m_\perp)] = \mp A(n_\perp, m_\perp) \cdot n_\perp + h^{n_\perp}[A(m_\perp)] \quad . \quad (306)$$

For $\bar{g}[A(\tilde{n}_\perp)]$ the expressions follow

$$\begin{aligned}\bar{g}[A(\tilde{n}_\perp)] &= \alpha \cdot \bar{g}[A(n_\perp)] + \beta \cdot \bar{g}[A(m_\perp)] = \\ &= \mp \alpha \cdot A(n_\perp, n_\perp) \cdot n_\perp + \alpha \cdot h^{n_\perp}[A(n_\perp)] \mp \\ &\quad \mp \beta \cdot A(n_\perp, m_\perp) \cdot n_\perp + \beta \cdot h^{n_\perp}[A(m_\perp)]\end{aligned}\quad (307)$$

$$\begin{aligned}\bar{g}[A(\tilde{n}_\perp)] &= \mp [\alpha \cdot A(n_\perp, n_\perp) + \beta \cdot A(n_\perp, m_\perp)] \cdot n_\perp + \\ &\quad + \alpha \cdot h^{n_\perp}[A(n_\perp)] + \beta \cdot h^{n_\perp}[A(m_\perp)] \quad .\end{aligned}\quad (308)$$

On the other side, the structures of $rel a = a_z + a_c$ could be represented under the condition $\mathcal{L}_u \xi_\perp = 0$ in the forms

$$rel a = l_{\xi_\perp} \cdot \bar{g}[A(n_\perp)] \quad , \quad (309)$$

$$a_z = \mp l_{\xi_\perp} \cdot A(n_\perp, n_\perp) \cdot n_\perp \quad , \quad (310)$$

$$\begin{aligned}a_c &= \bar{g}[h_{\xi_\perp}(rel a)] = l_{\xi_\perp} \cdot \bar{g}(h_{\xi_\perp})(\bar{g})[A(n_\perp)] = l_{\xi_\perp} \cdot h^{n_\perp}[A(n_\perp)] \\ g(a_c, n_\perp) &= 0 \quad ,\end{aligned}\quad (311)$$

$$rel a = a_z + a_c = l_{\xi_\perp} \cdot \bar{g}[A(n_\perp)] = \quad (312)$$

$$= \mp l_{\xi_\perp} \cdot A(n_\perp, n_\perp) \cdot n_\perp + l_{\xi_\perp} \cdot h^{n_\perp}[A(n_\perp)] \quad . \quad (313)$$

Let us introduce now a vector field $\eta_\perp = l_{\xi_\perp} \cdot m_\perp$, orthogonal to the vector field $\xi_\perp = l_{\xi_\perp} \cdot n_\perp$ and u , but with the same length as ξ_\perp , i.e.

$$g(\eta_\perp, \xi_\perp) = l_{\xi_\perp}^2 \cdot g(m_\perp, n_\perp) = 0 \quad , \quad g(\eta_\perp, \eta_\perp) = \mp l_{\xi_\perp}^2 \quad . \quad (314)$$

The corresponding to η_\perp relative velocity ${}_{rel}v_\eta$ and relative acceleration ${}_{rel}a_\eta$ have analogous forms as ${}_{rel}v$ and ${}_{rel}a$.

$${}_{rel}v_\eta = \overline{g}[d(\eta_\perp)] = l_{\xi_\perp} \cdot \overline{g}[d(m_\perp)] \quad , \quad (315)$$

$${}_{rel}a_\eta = \overline{g}[A(\eta_\perp)] = l_{\xi_\perp} \cdot \overline{g}[A(m_\perp)] \quad . \quad (316)$$

The decomposition of ${}_{rel}a_\eta$ has the form

$${}_{rel}a_\eta = a_{\eta z} + a_{\eta c} \quad , \quad (317)$$

$$\begin{aligned} {}_{rel}a_\eta &= \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot g({}_{rel}a_\eta, \xi_\perp) \cdot \xi_\perp + \overline{g}[h_{\xi_\perp}({}_{rel}a_\eta)] = a_{\eta z} + a_{\eta c} = \\ &= \mp g({}_{rel}a_\eta, n_\perp) \cdot n_\perp + \overline{g}[h_{\xi_\perp}({}_{rel}a_\eta)] \end{aligned} \quad (318)$$

where

$$a_{\eta z} = \mp g({}_{rel}a_\eta, n_\perp) \cdot n_\perp \quad , \quad (319)$$

$$a_{\eta c} = \overline{g}[h_{\xi_\perp}({}_{rel}a_\eta)] \quad . \quad (320)$$

The explicit form of $a_{\eta z}$ and $a_{\eta c}$ could be found by the use of the relations under the condition $\mathcal{L}_u \eta_\perp = 0$

$$\begin{aligned} g({}_{rel}a_\eta, n_\perp) &= g(\overline{g}[A(\eta_\perp)], n_\perp) = g_{ij} \cdot g^{ik} \cdot A_{kl} \cdot l_{\xi_\perp} \cdot m_\perp^l \cdot n_\perp^j = \\ &= l_{\xi_\perp} \cdot A_{jl} \cdot n_\perp^j \cdot m_\perp^l = l_{\xi_\perp} \cdot A(n_\perp, m_\perp) \quad , \end{aligned} \quad (321)$$

$$\overline{g}[h_{\xi_\perp}({}_{rel}a_\eta)] = l_{\xi_\perp} \cdot \overline{g}[h_{\xi_\perp}(\overline{g}[A(m_\perp)])] = l_{\xi_\perp} \cdot h^{n_\perp}[A(m_\perp)] \quad , \quad (322)$$

as

$$a_{\eta z} = \mp g({}_{rel}a_\eta, n_\perp) \cdot n_\perp = \mp l_{\xi_\perp} \cdot A(n_\perp, m_\perp) \cdot n_\perp \quad , \quad (323)$$

$$a_{\eta c} = \overline{g}[h_{\xi_\perp}({}_{rel}a_\eta)] = l_{\xi_\perp} \cdot h^{n_\perp}[A(m_\perp)] \quad . \quad (324)$$

Now we can find the relations between the relative velocities ${}_{rel}a$, ${}_{rel}a_\eta$, and the expression for $\overline{g}[A(\tilde{n}_\perp)]$

$$\begin{aligned} \overline{g}[A(\tilde{n}_\perp)] &= \alpha \cdot \overline{g}[A(n_\perp)] + \beta \cdot \overline{g}[A(m_\perp)] = \\ &= \mp \alpha \cdot A(n_\perp, n_\perp) \cdot n_\perp + \alpha \cdot h^{n_\perp}[A(n_\perp)] \mp \\ &\quad \mp \beta \cdot A(n_\perp, m_\perp) \cdot n_\perp + \beta \cdot h^{n_\perp}[A(m_\perp)] \quad , \end{aligned} \quad (325)$$

$$\begin{aligned} \overline{g}[A(\tilde{n}_\perp)] &= \mp \alpha \cdot (\mp \frac{1}{l_{\xi_\perp}} \cdot a_z) + \alpha \cdot \frac{1}{l_{\xi_\perp}} \cdot a_c \mp \\ &\quad \mp \beta \cdot (\mp \frac{1}{l_{\xi_\perp}} \cdot a_{\eta z}) + \beta \cdot \frac{1}{l_{\xi_\perp}} \cdot a_{\eta c} \quad , \end{aligned} \quad (326)$$

$$\begin{aligned}
\bar{g}[A(\tilde{n}_\perp)] &= \alpha \cdot \frac{1}{l_{\xi_\perp}} \cdot a_z + \alpha \cdot \frac{1}{l_{\xi_\perp}} \cdot a_c + \\
&+ \beta \cdot \frac{1}{l_{\xi_\perp}} \cdot a_{\eta z} + \beta \cdot \frac{1}{l_{\xi_\perp}} \cdot a_{\eta c} \quad , \quad (327)
\end{aligned}$$

$$\begin{aligned}
\bar{g}[A(\tilde{n}_\perp)] &= \frac{1}{l_{\xi_\perp}} \cdot [\alpha \cdot (a_z + a_c) + \beta \cdot (a_{\eta z} + a_{\eta c})] = \\
&= \frac{1}{l_{\xi_\perp}} \cdot (\alpha \cdot {}_{rel}a + \beta \cdot {}_{rel}a_\eta) \quad . \quad (328)
\end{aligned}$$

5. After the consideration and finding out of the explicit forms of the terms in ${}_{rel}k_\perp$

$$\begin{aligned}
{}_{rel}k_\perp &= -d\tau \cdot (\nabla_u \tilde{k})_\perp + \frac{1}{2} \cdot d\tau^2 \cdot (\nabla_u \nabla_u \tilde{k})_\perp = \\
&= -d\tau \cdot \left\{ \pm \frac{\omega}{l_u^2} \cdot a_\perp + \bar{g}[d(k_\perp)] \right\} + \\
&+ \frac{1}{2} \cdot d\tau^2 \cdot \left\{ \pm \frac{\omega}{l_u^2} \cdot (\nabla_u a)_\perp + \bar{g}[A(k_\perp)] \right\} \quad , \quad (329)
\end{aligned}$$

we can find the explicit forms of the change ${}_{rel}k_\perp$ of the vector k_\perp along the world line of the observer

$$\begin{aligned}
{}_{rel}k_\perp &= -d\tau \cdot \left\{ \pm \frac{\omega}{l_u^2} \cdot [(a_\perp)_z + (a_\perp)_c] \mp \right. \\
&\mp l_{k_\perp} \cdot \bar{g}[d(\tilde{n}_\perp)] \left. \right\} + \\
&+ \frac{1}{2} \cdot d\tau^2 \cdot \left\{ \pm \frac{\omega}{l_u^2} \cdot [(\nabla_u a)_\perp z + (\nabla_u a)_\perp c] \mp \right. \\
&\mp l_{k_\perp} \cdot \bar{g}[A(\tilde{n}_\perp)] \left. \right\} \quad , \quad (330)
\end{aligned}$$

$$\begin{aligned}
{}_{rel}k_\perp &= -d\tau \cdot \left\{ \pm \frac{\omega}{l_u^2} \cdot [(a_\perp)_z + (a_\perp)_c] \mp \right. \\
&\mp l_{k_\perp} \cdot \left[\frac{1}{l_{\xi_\perp}} \cdot (\alpha \cdot {}_{rel}v + \beta \cdot {}_{rel}v_\eta) \right] \left. \right\} + \\
&+ \frac{1}{2} \cdot d\tau^2 \cdot \left\{ \pm \frac{\omega}{l_u^2} \cdot [(\nabla_u a)_\perp z + (\nabla_u a)_\perp c] \mp \right. \\
&\mp l_{k_\perp} \cdot \left[\frac{1}{l_{\xi_\perp}} \cdot (\alpha \cdot {}_{rel}a + \beta \cdot {}_{rel}a_\eta) \right] \left. \right\} \quad , \quad (331)
\end{aligned}$$

$$\begin{aligned}
{}_{rel}k_\perp &= -d\tau \cdot \left\{ \pm \frac{\omega}{l_u^2} \cdot [(a_\perp)_z + (a_\perp)_c] \mp \right. \\
&\mp \frac{l_{k_\perp}}{l_{\xi_\perp}} \cdot (\alpha \cdot {}_{rel}v + \beta \cdot {}_{rel}v_\eta) \left. \right\} + \\
&+ \frac{1}{2} \cdot d\tau^2 \cdot \left\{ \pm \frac{\omega}{l_u^2} \cdot [(\nabla_u a)_\perp z + (\nabla_u a)_\perp c] \mp \right. \\
&\mp \frac{l_{k_\perp}}{l_{\xi_\perp}} \cdot (\alpha \cdot {}_{rel}a + \beta \cdot {}_{rel}a_\eta) \left. \right\} \quad . \quad (332)
\end{aligned}$$

Since

$$l_{k\perp} = \frac{\omega}{l_u} = l_{k\parallel} \quad , \quad (333)$$

we obtain the final form of $_{rel}k_{\perp}$ with respect to the relative velocity and relative acceleration

$$\begin{aligned} _{rel}k_{\perp} = & \mp d\tau \cdot l_{k\perp} \cdot \left\{ \frac{1}{l_u} \cdot [(a_{\perp})_z + (a_{\perp})_c] - \right. \\ & - \frac{1}{l_{\xi\perp}} \cdot (\alpha \cdot _{rel}v + \beta \cdot _{rel}v_{\eta}) \left. \right\} \pm \\ & \pm \frac{1}{2} \cdot d\tau^2 \cdot l_{k\perp} \cdot \left\{ \frac{1}{l_u} \cdot [(\nabla_u a)_{\perp z} + (\nabla_u a)_{\perp c}] - \right. \\ & - \frac{1}{l_{\xi\perp}} \cdot (\alpha \cdot _{rel}a + \beta \cdot _{rel}a_{\eta}) \left. \right\} \quad . \end{aligned} \quad (334)$$

If we, further, express the time interval $d\tau$ by its equivalent relations

$$d\tau = \mp \frac{l_{\xi\perp} \cdot d\lambda}{l_u} = \frac{dl}{l_u} \quad (335)$$

the relation of $_{rel}k_{\perp}$ to the relative velocity and relative acceleration could also be written in the forms

$$\begin{aligned} _{rel}k_{\perp} = & \frac{l_{k\perp}}{l_u} \cdot \left\{ \frac{l_{\xi\perp} \cdot d\lambda}{l_u} \cdot [(a_{\perp})_z + (a_{\perp})_c] - d\lambda \cdot (\alpha \cdot _{rel}v + \beta \cdot _{rel}v_{\eta}) \right\} \pm \\ & \pm \frac{1}{2} \cdot \frac{l_{\xi\perp}^2 \cdot d\lambda^2}{l_u^2} \cdot l_{k\perp} \cdot \left\{ \frac{1}{l_u} \cdot [(\nabla_u a)_{\perp z} + (\nabla_u a)_{\perp c}] - \right. \\ & - \frac{1}{l_{\xi\perp}} \cdot (\alpha \cdot _{rel}a + \beta \cdot _{rel}a_{\eta}) \left. \right\} \quad , \quad . \end{aligned} \quad (336)$$

$$\begin{aligned} _{rel}k_{\perp} = & \frac{l_{k\perp}}{l_u} \cdot \left\{ \mp \frac{dl}{l_u} \cdot [(a_{\perp})_z + (a_{\perp})_c] \pm \frac{dl}{l_{\xi\perp}} \cdot (\alpha \cdot _{rel}v + \beta \cdot _{rel}v_{\eta}) \right\} \pm \\ & \pm \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot l_{k\perp} \cdot \left\{ \frac{1}{l_u} \cdot [(\nabla_u a)_{\perp z} + (\nabla_u a)_{\perp c}] - \right. \\ & - \frac{1}{l_{\xi\perp}} \cdot (\alpha \cdot _{rel}a + \beta \cdot _{rel}a_{\eta}) \left. \right\} \quad . \end{aligned} \quad (337)$$

By the use of the explicit forms of \bar{k}_{\perp} and k_{\perp}

$$\bar{k}_{\perp} = \mp l_{\bar{k}\perp} \cdot \tilde{n}'_{\perp} = \mp \frac{\bar{\omega}}{l_u} \cdot \tilde{n}'_{\perp} \quad , \quad (338)$$

$$k_{\perp} = \mp l_{k\perp} \cdot \tilde{n}_{\perp} = \mp \frac{\omega}{l_u} \cdot \tilde{n}_{\perp} \quad , \quad (339)$$

we can find the explicit forms of the expressions

$$\bar{S} := \frac{g(_{rel}k_{\perp}, m_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})} = \frac{1}{l_{k\perp}} \cdot g(_{rel}k_{\perp}, m_{\perp}) \quad , \quad (340)$$

$$\bar{C} := \frac{g(_{rel}k_{\perp}, n_{\perp})}{g(\tilde{n}_{\perp}, k_{\perp})} = \frac{1}{l_{k\perp}} \cdot g(_{rel}k_{\perp}, n_{\perp}) \quad , \quad (341)$$

because of the relation

$$g(\tilde{n}_\perp, k_\perp) = g(\tilde{n}_\perp, \mp l_{k_\perp} \cdot \tilde{n}_\perp) = \mp l_{k_\perp} \cdot g(\tilde{n}_\perp, \tilde{n}_\perp) = l_{k_\perp} \quad . \quad (342)$$

6. By the use of the relations

$$g((a_\perp)_z + (a_\perp)_c, n_\perp) = g((a_\perp)_z, n_\perp) \quad , \quad g((a_\perp)_c, n_\perp) = 0 \quad , \quad (343)$$

$$g({}_{rel}v, n_\perp) = g(v_z, n_\perp) \quad , \quad g({}_{rel}v_\eta, n_\perp) = g(v_{\eta z}, n_\perp) \quad , \quad (344)$$

$$g((\nabla_u a)_{\perp z} + (\nabla_u a)_{\perp c}, n_\perp) = g((\nabla_u a)_{\perp z}, n_\perp) \quad , \quad (345)$$

$$g({}_{rel}a, n_\perp) = g(a_z, n_\perp) \quad , \quad g({}_{rel}a_\eta, n_\perp) = g(a_{\eta z}, n_\perp) \quad , \quad (346)$$

$$\begin{aligned} g({}_{rel}k_\perp, n_\perp) &= \frac{l_{k_\perp}}{l_u} \cdot \left\{ \mp \frac{dl}{l_u} \cdot g((a_\perp)_z, n_\perp) \pm \frac{dl}{l_{\xi_\perp}} \cdot [\alpha \cdot g(v_z, n_\perp) + \beta \cdot g(v_{\eta z}, n_\perp)] \right\} \pm \\ &\pm \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot l_{k_\perp} \cdot \left\{ \frac{1}{l_u} \cdot g((\nabla_u a)_{\perp z}, n_\perp) - \right. \\ &\left. - \frac{1}{l_{\xi_\perp}} \cdot [\alpha \cdot g(a_z, n_\perp) + \beta \cdot g(a_{\eta z}, n_\perp)] \right\} \quad . \end{aligned} \quad (347)$$

we can find the explicit form of the quantity \overline{C} .

On the other side, we can use the relations

$$(a_\perp)_z = \mp l_{(a_\perp)_z} \cdot n_\perp \quad , \quad (348)$$

$$g((a_\perp)_z, n_\perp) = \mp l_{(a_\perp)_z} \cdot g(n_\perp, n_\perp) = l_{(a_\perp)_z} \quad , \quad (349)$$

$$v_z = \mp l_{v_z} \cdot n_\perp \quad , \quad (350)$$

$$g(v_z, n_\perp) = \mp l_{v_z} \cdot g(n_\perp, n_\perp) = l_{v_z} \quad , \quad (351)$$

$$v_{\eta z} = \mp l_{v_{\eta z}} \cdot n_\perp \quad , \quad (352)$$

$$g(v_{\eta z}, n_\perp) = \mp l_{v_{\eta z}} \cdot g(n_\perp, n_\perp) = l_{v_{\eta z}} \quad , \quad (353)$$

$$(\nabla_u a)_{\perp z} = \mp l_{(\nabla_u a)_{\perp z}} \cdot n_\perp \quad , \quad (354)$$

$$g((\nabla_u a)_{\perp z}, n_\perp) = \mp l_{(\nabla_u a)_{\perp z}} \cdot g(n_\perp, n_\perp) = l_{(\nabla_u a)_{\perp z}} \quad , \quad (355)$$

$$g((\nabla_u a)_{\perp z}, m_\perp) = \mp l_{(\nabla_u a)_{\perp z}} \cdot g(n_\perp, m_\perp) = 0 \quad , \quad (356)$$

$$a_z = \mp l_{a_z} \cdot n_\perp \quad , \quad (357)$$

$$g(a_z, n_\perp) = \mp l_{a_z} \cdot g(n_\perp, n_\perp) = l_{a_z} \quad , \quad (358)$$

$$g(a_z, m_\perp) = \mp l_{a_z} \cdot g(n_\perp, m_\perp) = 0 \quad , \quad (359)$$

$$a_{\eta z} = \mp l_{a_{\eta z}} \cdot n_\perp \quad , \quad (360)$$

$$g(a_{\eta z}, n_\perp) = \mp l_{a_{\eta z}} \cdot g(n_\perp, n_\perp) = l_{a_{\eta z}} \quad , \quad (361)$$

$$g(a_{\eta z}, m_\perp) = \mp l_{a_{\eta z}} \cdot g(n_\perp, m_\perp) = 0 \quad , \quad (362)$$

for finding out the explicit form of $g(\text{rel}k_\perp, n_\perp)$ and $g(\text{rel}k_\perp, m_\perp)$.

$$g(\text{rel}k_\perp, n_\perp) = \frac{l_{k_\perp}}{l_u} \cdot \left\{ \mp \frac{dl}{l_u} \cdot l_{(a_\perp)_z} \pm \frac{dl}{l_{\xi_\perp}} \cdot [\alpha \cdot l_{v_z} + \beta \cdot l_{v_{\eta z}}] \right\} \pm \quad (363)$$

$$\pm \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot l_{k_\perp} \cdot \left\{ \frac{1}{l_u} \cdot l_{(\nabla_u a)_{\perp z}} - \frac{1}{l_{\xi_\perp}} \cdot [\alpha \cdot l_{a_z} + \beta \cdot l_{a_{\eta z}}] \right\} \quad . \quad (364)$$

or the form

$$\begin{aligned} g(\text{rel}k_\perp, n_\perp) &= \frac{l_{k_\perp}}{l_u} \cdot \left\{ \pm \frac{dl}{l_{\xi_\perp}} \cdot [\alpha \cdot l_{v_z} + \beta \cdot l_{v_{\eta z}}] \mp \frac{dl}{l_u} \cdot l_{(a_\perp)_z} \right\} \mp \\ &\mp \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot l_{k_\perp} \cdot \left\{ \frac{1}{l_{\xi_\perp}} \cdot [\alpha \cdot l_{a_z} + \beta \cdot l_{a_{\eta z}}] - \frac{1}{l_u} \cdot l_{(\nabla_u a)_{\perp z}} \right\} \end{aligned} \quad (365)$$

The explicit form of \overline{C} could now be found as

$$\begin{aligned} \overline{C} &= \frac{1}{l_{k_\perp}} \cdot g(\text{rel}k_\perp, n_\perp) = \\ &= \pm \frac{1}{l_u} \cdot \left\{ \frac{dl}{l_{\xi_\perp}} \cdot [\alpha \cdot l_{v_z} + \beta \cdot l_{v_{\eta z}}] - \frac{dl}{l_u} \cdot l_{(a_\perp)_z} \right\} \mp \\ &\mp \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot \left\{ \frac{1}{l_{\xi_\perp}} \cdot [\alpha \cdot l_{a_z} + \beta \cdot l_{a_{\eta z}}] - \frac{1}{l_u} \cdot l_{(\nabla_u a)_{\perp z}} \right\} . \end{aligned} \quad (366)$$

If we introduce the abbreviations

$$\bar{l}_{v_z} = \frac{dl}{l_{\xi_\perp}} \cdot l_{v_z} \quad , \quad \bar{l}_{v_{\eta z}} = \frac{dl}{l_{\xi_\perp}} \cdot l_{v_{\eta z}} \quad , \quad \bar{l}_{(a_\perp)_z} = \frac{dl}{l_{\xi_\perp}} \cdot l_{(a_\perp)_z} \quad , \quad (367)$$

$$\bar{l}_{a_z} = \frac{dl}{l_{\xi_\perp}} \cdot l_{a_z} \quad , \quad \bar{l}_{a_{\eta z}} = \frac{dl}{l_{\xi_\perp}} \cdot l_{a_{\eta z}} \quad , \quad \bar{l}_{(\nabla_u a)_{\perp z}} = \frac{dl}{l_{\xi_\perp}} \cdot l_{(\nabla_u a)_{\perp z}} \quad , \quad (368)$$

then \overline{C} could be represented in the form

$$\begin{aligned} \overline{C} &= \pm \frac{1}{l_u} \cdot [(\alpha \cdot \bar{l}_{v_z} + \beta \cdot \bar{l}_{v_{\eta z}}) - \frac{l_{\xi_\perp}}{l_u} \cdot \bar{l}_{(a_\perp)_z}] \mp \\ &\mp \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot [(\alpha \cdot \bar{l}_{a_z} + \beta \cdot \bar{l}_{a_{\eta z}}) - \frac{l_{\xi_\perp}}{l_u} \cdot \bar{l}_{(\nabla_u a)_{\perp z}}] . \end{aligned} \quad (369)$$

In analogous way, we can find the explicit form of \overline{S} .

7. By the use of the relations

$$\begin{aligned} g(\text{rel}k_\perp, m_\perp) &= \frac{l_{k_\perp}}{l_u} \cdot \left\{ \mp \frac{dl}{l_u} \cdot [g((a_\perp)_z, m_\perp) + g((a_\perp)_c, m_\perp)] \pm \right. \\ &\pm \frac{dl}{l_{\xi_\perp}} \cdot [\alpha \cdot g(\text{rel}v, m_\perp) + \beta \cdot g(\text{rel}v_\eta, m_\perp)] \left. \right\} \pm \\ &\pm \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot l_{k_\perp} \cdot \left\{ \frac{1}{l_u} \cdot [g((\nabla_u a)_{\perp z}, m_\perp) + g((\nabla_u a)_{\perp c}, m_\perp)] - \right. \\ &\left. - \frac{1}{l_{\xi_\perp}} \cdot [\alpha \cdot g(\text{rel}a, m_\perp) + \beta \cdot g(\text{rel}a_\eta, m_\perp)] \right\} \quad , \end{aligned} \quad (370)$$

$$\begin{aligned}
g(\textit{rel}k_{\perp}, m_{\perp}) &= \frac{l_{k_{\perp}}}{l_u} \cdot \left\{ \mp \frac{dl}{l_u} \cdot g((a_{\perp})_c, m_{\perp}) \pm \right. \\
&\quad \left. \pm \frac{dl}{l_{\xi_{\perp}}} \cdot [\alpha \cdot g(\textit{rel}v, m_{\perp}) + \beta \cdot g(\textit{rel}v_{\eta}, m_{\perp})] \right\} \pm \\
&\quad \pm \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot l_{k_{\perp}} \cdot \left\{ \frac{1}{l_u} \cdot g((\nabla_u a)_{\perp c}, m_{\perp}) - \right. \\
&\quad \left. - \frac{1}{l_{\xi_{\perp}}} \cdot [\alpha \cdot g(\textit{rel}a, m_{\perp}) + \beta \cdot g(\textit{rel}a_{\eta}, m_{\perp})] \right\} \quad , \quad (371)
\end{aligned}$$

$$(a_{\perp})_c = \mp l_{(a_{\perp})_c} \cdot m_{\perp} \quad , \quad (372)$$

$$g((a_{\perp})_c, m_{\perp}) = \mp l_{(a_{\perp})_c} \cdot g(m_{\perp}, m_{\perp}) = l_{(a_{\perp})_c} \quad , \quad (373)$$

$$v_c = \mp l_{v_c} \cdot m_{\perp} \quad , \quad (374)$$

$$\begin{aligned}
g(\textit{rel}v, m_{\perp}) &= g(v_z + v_c, m_{\perp}) = \\
&= g(v_c, m_{\perp}) = \mp l_{v_c} \cdot g(m_{\perp}, m_{\perp}) = l_{v_c} \quad , \quad (375)
\end{aligned}$$

$$v_{\eta c} = \mp l_{v_{\eta c}} \cdot m_{\perp} \quad , \quad (376)$$

$$\begin{aligned}
g(\textit{rel}v_{\eta}, m_{\perp}) &= g(v_{\eta z} + v_{\eta c}, m_{\perp}) = g(v_{\eta c}, m_{\perp}) = \mp l_{v_{\eta c}} \cdot g(m_{\perp}, m_{\perp}) \\
&= l_{v_{\eta c}} \quad , \quad (378)
\end{aligned}$$

$$(\nabla_u a)_{\perp c} = \mp l_{(\nabla_u a)_{\perp c}} \cdot m_{\perp} \quad , \quad (379)$$

$$g((\nabla_u a)_{\perp c}, m_{\perp}) = \mp l_{(\nabla_u a)_{\perp c}} \cdot g(m_{\perp}, m_{\perp}) = l_{(\nabla_u a)_{\perp c}} \quad , \quad (380)$$

$$\textit{rel}a = a_z + a_c \quad , \quad a_z = \mp l_{a_z} \cdot n_{\perp} \quad , \quad a_c = \mp l_{a_c} \cdot m_{\perp} \quad , \quad (381)$$

$$g(\textit{rel}a, m_{\perp}) = g(a_z + a_c, m_{\perp}) = g(a_c, m_{\perp}) = \mp l_{a_c} \cdot g(m_{\perp}, m_{\perp}) = l_{a_c} \quad , \quad (382)$$

$$\textit{rel}a_{\eta} = a_{\eta z} + a_{\eta c} \quad , \quad a_{\eta z} = \mp l_{a_{\eta z}} \cdot n_{\perp} \quad , \quad a_{\eta c} = \mp l_{a_{\eta c}} \cdot m_{\perp} \quad , \quad (383)$$

$$g(\textit{rel}a_{\eta}, m_{\perp}) = g(a_{\eta c}, m_{\perp}) = \mp l_{a_{\eta c}} \cdot g(m_{\perp}, m_{\perp}) = l_{a_{\eta c}} \quad . \quad (384)$$

we can find the explicit forms of $g(\textit{rel}v_{\eta}, m_{\perp})$ related to the centrifugal (centripetal) and Coriolis velocities and accelerations:

$$\begin{aligned}
g(\textit{rel}k_{\perp}, m_{\perp}) &= \frac{l_{k_{\perp}}}{l_u} \cdot \left\{ \mp \frac{dl}{l_u} \cdot l_{(a_{\perp})_c} \pm \frac{dl}{l_{\xi_{\perp}}} \cdot [\alpha \cdot l_{v_c} + \beta \cdot l_{v_{\eta c}}] \right\} \pm \\
&\quad \pm \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot l_{k_{\perp}} \cdot \left\{ \frac{1}{l_u} \cdot l_{(\nabla_u a)_{\perp c}} - \frac{1}{l_{\xi_{\perp}}} \cdot [\alpha \cdot l_{a_c} + \beta \cdot l_{a_{\eta c}}] \right\} \quad , \quad (385) \\
g(\textit{rel}k_{\perp}, m_{\perp}) &= \pm \frac{l_{k_{\perp}}}{l_u} \cdot \left[\frac{dl}{l_{\xi_{\perp}}} \cdot (\alpha \cdot l_{v_c} + \beta \cdot l_{v_{\eta c}}) - \frac{dl}{l_u} \cdot l_{(a_{\perp})_c} \right] \mp
\end{aligned}$$

$$\mp \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot l_{k_\perp} \cdot \left[\frac{1}{l_{\xi_\perp}} \cdot (\alpha \cdot l_{a_c} + \beta \cdot l_{a_{\eta c}}) - \frac{1}{l_u} \cdot l_{(\nabla_u a)_\perp c} \right] . \quad (386)$$

The explicit form of \bar{S} could be found as

$$\begin{aligned} \bar{S} & : = \frac{1}{l_{k_\perp}} \cdot g_{(rel)k_\perp, m_\perp} = \\ & = \pm \frac{1}{l_u} \cdot \left[\frac{dl}{l_{\xi_\perp}} \cdot (\alpha \cdot l_{v_c} + \beta \cdot l_{v_{\eta c}}) - \frac{dl}{l_u} \cdot l_{(a_\perp)_c} \right] \mp \\ & \mp \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot \left[\frac{1}{l_{\xi_\perp}} \cdot (\alpha \cdot l_{a_c} + \beta \cdot l_{a_{\eta c}}) - \frac{1}{l_u} \cdot l_{(\nabla_u a)_\perp c} \right] . \end{aligned} \quad (387)$$

If we introduce the abbreviations

$$\bar{l}_{v_c} = \frac{dl}{l_{\xi_\perp}} \cdot l_{v_c} , \quad \bar{l}_{v_{\eta c}} = \frac{dl}{l_{\xi_\perp}} \cdot l_{v_{\eta c}} , \quad \bar{l}_{(a_\perp)_c} = \frac{dl}{l_{\xi_\perp}} \cdot l_{(a_\perp)_c} , \quad (388)$$

$$\bar{l}_{a_c} = \frac{dl}{l_{\xi_\perp}} \cdot l_{a_c} , \quad \bar{l}_{a_{\eta c}} = \frac{dl}{l_{\xi_\perp}} \cdot l_{a_{\eta c}} , \quad \bar{l}_{(\nabla_u a)_\perp c} = \frac{dl}{l_{\xi_\perp}} \cdot l_{(\nabla_u a)_\perp c} \quad (389)$$

$$\bar{l}_\diamond \leq 0 . \quad (390)$$

then \bar{S} could be represented in the form

$$\begin{aligned} \bar{S} & = \pm \frac{1}{l_u} \cdot [(\alpha \cdot \bar{l}_{v_c} + \beta \cdot \bar{l}_{v_{\eta c}}) - \frac{l_{\xi_\perp}}{l_u} \cdot \bar{l}_{(a_\perp)_c}] \mp \\ & \mp \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot [(\alpha \cdot \bar{l}_{a_c} + \beta \cdot \bar{l}_{a_{\eta c}}) - \frac{l_{\xi_\perp}}{l_u} \cdot \bar{l}_{(\nabla_u a)_\perp c}] . \end{aligned} \quad (391)$$

The explicit forms of \bar{C} and \bar{S} determine the relations describing the aberration, the Doppler effect, and the Hubble effect in spaces with affine connections and metrics.

5 Aberration

Aberration is the deviation of the direction of the vector field \bar{k}_\perp from the direction of the vector field k_\perp . If $\bar{k}_\perp = \mp l_{\bar{k}_\perp} \cdot \tilde{n}'_\perp$ and $k_\perp = \mp l_{k_\perp} \cdot \tilde{n}_\perp$ then the difference between the angles θ' for \bar{k}_\perp and θ for k_\perp with respect to the direction of the vector field ξ_\perp is given by the relations

$$\begin{aligned} \frac{\bar{\omega}}{\omega} \cdot \cos \theta' & = \cos \theta + \frac{1}{l_{k_\perp}} \cdot g_{(rel)k_\perp, n_\perp} = \\ & = \cos \theta + \bar{C} , \end{aligned} \quad (392)$$

$$\begin{aligned} \frac{\bar{\omega}}{\omega} \cdot \sin \theta' & = \sin \theta + \frac{1}{l_{k_\perp}} \cdot g_{(rel)k_\perp, m_\perp} = \\ & = \sin \theta + \bar{S} . \end{aligned} \quad (393)$$

From the last (above) two relations, it follows for $tg\theta'$

$$\begin{aligned} tg\theta' &= \frac{\sin\theta'}{\cos\theta'} = \frac{\sin\theta + \frac{1}{l_{k\perp}} \cdot g(\text{rel}k_{\perp}, m_{\perp})}{\cos\theta + \frac{1}{l_{k\perp}} \cdot g(\text{rel}k_{\perp}, n_{\perp})} = \\ &= \frac{\sin\theta + \overline{S}}{\cos\theta + \overline{C}} . \end{aligned} \quad (394)$$

If there is no relative velocities and no relative accelerations between the emitter and the observer then $\overline{C} = 0$ and $\overline{S} = 0$. Then

$$\frac{\overline{\omega}}{\omega} \cdot \cos\theta' = \cos\theta , \quad (395)$$

$$\frac{\overline{\omega}}{\omega} \cdot \sin\theta' = \sin\theta , \quad (396)$$

$$\begin{aligned} \frac{\overline{\omega}^2}{\omega^2} &= 1 , \\ \overline{\omega} &= \omega . \end{aligned} \quad (397)$$

The frequency of the signal, emitted by the emitter, and the frequency of the signal, detected by the observer, are the same and, at the same time, no aberration occurs.

Special case: Auto-parallel motion ($\nabla_u u = a = 0$) of an observer detected a signal with emitted frequency $\overline{\omega}$. Then

$$\overline{C} = \pm \frac{1}{l_u} \cdot [(\alpha \cdot \overline{l}_{v_z} + \beta \cdot \overline{l}_{v_{\eta z}})] \mp \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot [(\alpha \cdot \overline{l}_{a_z} + \beta \cdot \overline{l}_{a_{\eta z}})] . \quad (398)$$

$$\overline{S} = \pm \frac{1}{l_u} \cdot [(\alpha \cdot \overline{l}_{v_c} + \beta \cdot \overline{l}_{v_{\eta c}})] \mp \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot [(\alpha \cdot \overline{l}_{a_c} + \beta \cdot \overline{l}_{a_{\eta c}})] . \quad (399)$$

$$\overline{\omega} \cdot \sin\theta' = \{ \sin\theta \pm \frac{1}{l_u} \cdot [(\alpha \cdot \overline{l}_{v_c} + \beta \cdot \overline{l}_{v_{\eta c}})] \mp \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot [(\alpha \cdot \overline{l}_{a_c} + \beta \cdot \overline{l}_{a_{\eta c}})] \} \cdot \omega , \quad (400)$$

$$\overline{\omega} \cdot \cos\theta' = \{ \cos\theta \pm \frac{1}{l_u} \cdot [(\alpha \cdot \overline{l}_{v_z} + \beta \cdot \overline{l}_{v_{\eta z}})] \mp \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot [(\alpha \cdot \overline{l}_{a_z} + \beta \cdot \overline{l}_{a_{\eta z}})] \} \cdot \omega , \quad (401)$$

$$\alpha = \mp \cos\theta , \quad \beta = \mp \sin\theta . \quad (402)$$

6 Doppler effect

The *Doppler effect* (Doppler shift) is the shift of signal's frequency caused by the relative motion between the emitter and the observer.

1. Usually, in classical mechanics, and especially in acoustics, there is a difference between the shift of the frequency when the observer is moving to or out of the emitter and the shift of the frequency when the emitter is moving to or out of the observer. In the first case, the signal is propagating in a medium at

rest with respect to the emitter, and in the second case, the signal is propagating in a medium moving with respect to the emitter. It is assumed that the signal is propagating in a continuous media used as a carrier of the signal.

In relativistic physics, and especially in electrodynamics, there is no difference between the shifts of the frequency of a signal when the observer is moving to or out of the emitter and when the emitter is moving to or out of the observer. The relative motion is the only reason for the shift frequency.

2. If the shift of the frequency of a signal is considered from the point of view of an observer then only the relative motions with respect to the observer could be taken into account. The observer detects the signals in his proper frame of reference (laboratory) and could make a comparison between the signals sent by emitter at rest with respect to his proper frame of reference and by emitter moving relatively to observer's proper frame of reference.

3. The observer could move in space-time where the space could be filled with a continuous media or with classical fields with physical interpretation. Since every classical field theory could be considered as a theory of a continuous media [1], [2], [4] both type of theories could be used for dynamical description of propagation of signals in space-time. An observer is interested in finding out how signals are propagating, how the emitter are moving with respect to the observer, and how the signals are generated by an emitter. Only the first two questions are subjects of consideration by the use of kinematic characteristics of the relative velocity, the relative acceleration, and the properties of null (isotropic) vector fields. The last question is a matter of considerations of the corresponding dynamical theory.

4. The Doppler effect could be described in spaces with affine connections and metrics as models of space or space-time on the basis of the relations between the emitted frequency $\bar{\omega}$ and detected frequency ω of signals propagating in space or space-time. The same relations are used for consideration of the aberration of signals. As corollary of them a relation between $\bar{\omega}$ and ω follows in the form

$$\bar{\omega} = [(\sin \theta + \bar{S})^2 + (\cos \theta + \bar{C})^2]^{1/2} \cdot \omega \quad (403)$$

appearing as a general formula for the generalized Doppler effect in spaces with affine connections and metrics.

6.1 Standard (longitudinal) Doppler effect (Doppler shift)

1. The standard (longitudinal) Doppler effect appears when all Coriolis velocities and Coriolis accelerations are compensating each other or do not exist in the relative motion between emitter and detector (observer), i.e. if

$$\bar{S} = 0 \quad . \quad (404)$$

Then

$$\begin{aligned} \bar{\omega} &= [(\sin^2 \theta + (\cos \theta + \bar{C})^2)^{1/2} \cdot \omega = \\ &= [\sin^2 \theta + \cos^2 \theta + 2 \cdot \bar{C} \cdot \cos \theta + \bar{C}^2]^{1/2} \cdot \omega \end{aligned} \quad (405)$$

$$\bar{\omega} = [1 + 2 \cdot \bar{C} \cdot \cos \theta + \bar{C}^2]^{1/2} \cdot \omega \quad , \quad (406)$$

where

$$\begin{aligned}
\overline{C} &= \pm \frac{1}{l_u} \cdot [(\mp \cos \theta \cdot \bar{l}_{v_z} \mp \sin \theta \cdot \bar{l}_{v_{\eta z}}) - \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(a_{\perp})_z}] \mp \\
&\quad \mp \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot [(\mp \cos \theta \cdot \bar{l}_{a_z} \mp \sin \theta \cdot \bar{l}_{a_{\eta z}}) - \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(\nabla_u a)_{\perp z}}] , \\
\overline{C} &= \frac{1}{l_u} \cdot [-(\cos \theta \cdot \bar{l}_{v_z} + \sin \theta \cdot \bar{l}_{v_{\eta z}}) \mp \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(a_{\perp})_z}] + \\
&\quad + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot [(\cos \theta \cdot \bar{l}_{a_z} + \sin \theta \cdot \bar{l}_{a_{\eta z}}) \pm \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(\nabla_u a)_{\perp z}}] . \quad (407)
\end{aligned}$$

2. If the vector field k_{\perp} is collinear to the vector field ξ_{\perp} determining the proper frame of reference, i.e. if

$$k_{\perp} = \mp l_{k_{\perp}} \cdot n_{\perp} , \quad \cos \theta = \pm 1 , \quad \sin \theta = 0 , \quad (408)$$

then

$$\begin{aligned}
\overline{C} &= \frac{1}{l_u} \cdot (\mp \bar{l}_{v_z} \mp \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(a_{\perp})_z}) + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot (\pm \bar{l}_{a_z} \pm \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(\nabla_u a)_{\perp z}}) , \\
\overline{C} &= \mp [\frac{1}{l_u} \cdot (\bar{l}_{v_z} + \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(a_{\perp})_z}) - \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot (\bar{l}_{a_z} + \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(\nabla_u a)_{\perp z}})] , \quad (409)
\end{aligned}$$

$$\begin{aligned}
\overline{\omega} &= [1 \pm 2 \cdot \overline{C} + \overline{C}^2]^{1/2} \cdot \omega = \\
&= [(1 \pm \overline{C})^2]^{1/2} \cdot \omega = (1 \pm \overline{C}) \cdot \omega , \quad (410)
\end{aligned}$$

$$\frac{\overline{\omega} - \omega}{\omega} = \pm \overline{C} = z , \quad (411)$$

$$\begin{aligned}
\overline{\omega} &= \omega \cdot [1 - \frac{1}{l_u} \cdot (\bar{l}_{v_z} + \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(a_{\perp})_z}) + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot (\bar{l}_{a_z} + \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(\nabla_u a)_{\perp z}})] = \\
&= \omega \cdot [1 - \frac{\bar{l}_{v_z}}{l_u} - \frac{l_{\xi_{\perp}}}{l_u^2} \cdot \bar{l}_{(a_{\perp})_z} + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot (\bar{l}_{a_z} + \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(\nabla_u a)_{\perp z}})] . \quad (412)
\end{aligned}$$

Special case: Auto-parallel motion of the observer: $\nabla_u u = a = 0$: $\bar{l}_{(a_{\perp})_z} = 0$, $\bar{l}_{(\nabla_u a)_{\perp z}} = 0$,

$$\begin{aligned}
\overline{\omega} &= \omega \cdot [1 - \frac{\bar{l}_{v_z}}{l_u} + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot \bar{l}_{a_z}] , \quad (413) \\
\bar{l}_{v_z} &\leq 0 , \quad \bar{l}_{a_z} \leq 0 .
\end{aligned}$$

If the world line of an observer is an auto-parallel trajectory and k_{\perp} is collinear to ξ_{\perp} then the change of the frequency $\overline{\omega}$ of the emitter depends on the centrifugal (centripetal) velocity \bar{l}_{v_z} and the centrifugal (centripetal) acceleration \bar{l}_{a_z} .

Special case: $\nabla_u u = a = 0$, $k_\perp = \mp l_{k_\perp} \cdot n_\perp, \bar{l}_{a_z} = 0$:

$$\bar{\omega} = \omega \cdot \left(1 - \frac{\bar{l}_{v_z}}{l_u}\right) \quad , \quad \bar{l}_{v_z} \leq 0 \quad . \quad (414)$$

Therefore, if the world line of an observer is an auto-parallel trajectory, k_\perp is collinear to ξ_\perp , and no centrifugal (centripetal) acceleration \bar{l}_{a_z} exists between emitter and observer then the above expression has the well known form for description of the standard Doppler effect in classical mechanics in 3-dimensional Euclidean space. Here, the relation is valid in every (\bar{L}_n, g) -space considered as a model of a space or a space-time under the given preconditions.

6.2 Transversal Doppler effect

1. The transversal Doppler effect appears when all centrifugal (centripetal) velocities and centrifugal (centripetal) accelerations are compensating each other or do not exist in the relative motion between emitter and detector (observer), i.e. if

$$\bar{C} = 0 \quad . \quad (415)$$

Then

$$\begin{aligned} \bar{\omega} &= [(\sin \theta + \bar{S})^2 + \cos^2 \theta]^{1/2} \cdot \omega = \\ &= [\sin^2 \theta + 2 \cdot \bar{S} \cdot \sin \theta + \bar{S}^2 + \cos^2 \theta]^{1/2} \cdot \omega = \\ &= [1 + 2 \cdot \bar{S} \cdot \sin \theta + \bar{S}^2]^{1/2} \cdot \omega \quad , \end{aligned} \quad (416)$$

where

$$\begin{aligned} \bar{S} : &= \pm \frac{1}{l_u} \cdot \left[\frac{dl}{l_{\xi_\perp}} \cdot (\alpha \cdot l_{v_c} + \beta \cdot l_{v_{\eta c}}) - \frac{dl}{l_u} \cdot l_{(a_\perp)_c} \right] \mp \\ &\mp \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot \left[\frac{1}{l_{\xi_\perp}} \cdot (\alpha \cdot l_{a_c} + \beta \cdot l_{a_{\eta c}}) - \frac{1}{l_u} \cdot l_{(\nabla_u a)_\perp c} \right] \quad , \end{aligned} \quad (417)$$

$$\alpha = \mp \cos \theta \quad , \quad \beta = \mp \sin \theta \quad , \quad (418)$$

$$\begin{aligned} \bar{S} : &= \pm \frac{1}{l_u} \cdot \left[\frac{dl}{l_{\xi_\perp}} \cdot (\mp \cos \theta \cdot l_{v_c} \mp \sin \theta \cdot l_{v_{\eta c}}) - \frac{dl}{l_u} \cdot l_{(a_\perp)_c} \right] \mp \\ &\mp \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot \left[\frac{1}{l_{\xi_\perp}} \cdot (\mp \cos \theta \cdot l_{a_c} \mp \sin \theta \cdot l_{a_{\eta c}}) - \frac{1}{l_u} \cdot l_{(\nabla_u a)_\perp c} \right] \quad , \end{aligned} \quad (419)$$

$$\begin{aligned} \bar{S} : &= \frac{1}{l_u} \cdot \left[-\frac{dl}{l_{\xi_\perp}} \cdot (\cos \theta \cdot l_{v_c} + \sin \theta \cdot l_{v_{\eta c}}) \mp \frac{dl}{l_u} \cdot l_{(a_\perp)_c} \right] + \\ &+ \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot \left[\frac{1}{l_{\xi_\perp}} \cdot (\cos \theta \cdot l_{a_c} + \sin \theta \cdot l_{a_{\eta c}}) \pm \frac{1}{l_u} \cdot l_{(\nabla_u a)_\perp c} \right] \quad . \end{aligned} \quad (420)$$

2. If the vector field k_\perp is orthogonal to the vector field ξ_\perp determining the proper frame of reference, i.e. if

$$k_\perp = \mp l_{k_\perp} \cdot m_\perp \quad , \quad \sin \theta = \pm 1 \quad , \quad \cos \theta = 0 \quad , \quad (421)$$

then

$$\begin{aligned}
\bar{\omega} &= [1 + 2 \cdot \bar{S} \cdot \sin\theta + \bar{S}^2]^{1/2} \cdot \omega = \\
&= [1 \pm 2 \cdot \bar{S} + \bar{S}^2]^{1/2} \cdot \omega = \\
&= (1 \pm \bar{S}) \cdot \omega \quad ,
\end{aligned} \tag{422}$$

$$\frac{\bar{\omega} - \omega}{\omega} = \pm \bar{S} = z_c \quad , \tag{423}$$

$$\begin{aligned}
\bar{S} : &= \frac{1}{l_u} \cdot \left[-\frac{dl}{l_{\xi_{\perp}}} \cdot (\pm l_{v_{\eta c}}) \mp \frac{dl}{l_u} \cdot l_{(a_{\perp})_c} \right] + \\
&+ \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot \left(\frac{1}{l_{\xi_{\perp}}} \cdot (\pm l_{a_{\eta c}}) \pm \frac{1}{l_u} \cdot l_{(\nabla_u a)_{\perp c}} \right) \quad ,
\end{aligned}$$

$$\begin{aligned}
\bar{S} : &= \frac{1}{l_u} \cdot \left(\mp \frac{dl}{l_{\xi_{\perp}}} \cdot l_{v_{\eta c}} \mp \frac{dl}{l_u} \cdot l_{(a_{\perp})_c} \right) + \\
&+ \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot \left(\pm \frac{1}{l_{\xi_{\perp}}} \cdot l_{a_{\eta c}} \pm \frac{1}{l_u} \cdot l_{(\nabla_u a)_{\perp c}} \right) \quad ,
\end{aligned}$$

$$\begin{aligned}
\bar{S} : &= \mp \frac{1}{l_u} \cdot \left(\frac{dl}{l_{\xi_{\perp}}} \cdot l_{v_{\eta c}} + \frac{dl}{l_u} \cdot l_{(a_{\perp})_c} \right) \pm \\
&\pm \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot \left(\frac{1}{l_{\xi_{\perp}}} \cdot l_{a_{\eta c}} + \frac{1}{l_u} \cdot l_{(\nabla_u a)_{\perp c}} \right) \quad ,
\end{aligned} \tag{424}$$

$$\begin{aligned}
\bar{\omega} &= \left\{ 1 - \frac{1}{l_u} \cdot \left(\frac{dl}{l_{\xi_{\perp}}} \cdot l_{v_{\eta c}} + \frac{dl}{l_u} \cdot l_{(a_{\perp})_c} \right) + \right. \\
&\left. + \frac{1}{2} \cdot \frac{dl^2}{l_u^2} \cdot \left(\frac{1}{l_{\xi_{\perp}}} \cdot l_{a_{\eta c}} + \frac{1}{l_u} \cdot l_{(\nabla_u a)_{\perp c}} \right) \right\} \cdot \omega \quad .
\end{aligned} \tag{425}$$

If we introduce the abbreviations

$$\bar{l}_{v_{\eta c}} = \frac{dl}{l_{\xi_{\perp}}} \cdot l_{v_{\eta c}} \quad , \quad \bar{l}_{(a_{\perp})_c} = \frac{dl}{l_{\xi_{\perp}}} \cdot l_{(a_{\perp})_c} \quad , \tag{426}$$

$$\bar{l}_{a_{\eta c}} = \frac{dl}{l_{\xi_{\perp}}} \cdot l_{a_{\eta c}} \quad , \quad \bar{l}_{(\nabla_u a)_{\perp c}} = \frac{dl}{l_{\xi_{\perp}}} \cdot l_{(\nabla_u a)_{\perp c}} \quad , \tag{427}$$

$$\bar{l}_{\diamond} \begin{matrix} \leq \\ \geq \end{matrix} 0 \quad ,$$

then the expressions for \bar{S} and $\bar{\omega}$ will have the forms

$$\begin{aligned}
\bar{S} &= \mp \frac{1}{l_u} \cdot (\bar{l}_{v_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(a_{\perp})_c}) \pm \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot (\bar{l}_{a_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \cdot l_{(\nabla_u a)_{\perp c}}) = \\
&= \mp \left[\frac{1}{l_u} \cdot (\bar{l}_{v_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(a_{\perp})_c}) - \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot (\bar{l}_{a_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \cdot l_{(\nabla_u a)_{\perp c}}) \right] \quad ,
\end{aligned}$$

$$\begin{aligned}
\pm \bar{S} &= -\left[\frac{1}{l_u} \cdot (\bar{l}_{v_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(a_{\perp})_c}) - \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot (\bar{l}_{a_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \cdot l_{(\nabla_u a)_{\perp c}})\right] = \\
&= -\frac{1}{l_u} \cdot (\bar{l}_{v_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(a_{\perp})_c}) + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot (\bar{l}_{a_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \cdot l_{(\nabla_u a)_{\perp c}}) , \quad (428)
\end{aligned}$$

$$\begin{aligned}
\bar{\omega} &= \omega \cdot \left[1 - \frac{1}{l_u} \cdot (\bar{l}_{v_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(a_{\perp})_c}) + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot (\bar{l}_{a_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \cdot l_{(\nabla_u a)_{\perp c}})\right] = \\
&= \omega \cdot \left[1 - \frac{1}{l_u} \cdot \bar{l}_{v_{\eta c}} - \frac{l_{\xi_{\perp}}}{l_u^2} \cdot \bar{l}_{(a_{\perp})_c} + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot (\bar{l}_{a_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \cdot l_{(\nabla_u a)_{\perp c}})\right] . \quad (429)
\end{aligned}$$

Remark. The expression for \bar{S} has the same form as the expression for \bar{C} under the change of \bar{l}_{v_z} with $\bar{l}_{v_{\eta c}}$, $\bar{l}_{(a_{\perp})_z}$ with $\bar{l}_{(a_{\perp})_c}$, and $\bar{l}_{(\nabla_u a)_{\perp z}}$ with $l_{(\nabla_u a)_{\perp c}}$.

Special case: Auto-parallel motion of the observer: $\nabla_u u = a = 0$: $\bar{l}_{(a_{\perp})_c} = 0$, $\bar{l}_{(\nabla_u a)_{\perp c}} = 0$,

$$\bar{\omega} = \omega \cdot \left[1 - \frac{1}{l_u} \cdot \bar{l}_{v_{\eta c}} + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot \bar{l}_{a_{\eta c}}\right] . \quad (430)$$

If the world line of an observer is an auto-parallel trajectory and k_{\perp} is orthogonal to ξ_{\perp} then the change of the frequency $\bar{\omega}$ of the emitter depends on the Coriolis velocity $\bar{l}_{v_{\eta c}}$ and the Coriolis acceleration $\bar{l}_{a_{\eta c}}$.

Special case: $\nabla_u u = a = 0$, $k_{\perp} = \mp l_{k_{\perp}} \cdot m_{\perp}$, $\bar{l}_{a_{\eta c}} = 0$:

$$\bar{\omega} = \omega \cdot \left(1 - \frac{\bar{l}_{v_{\eta c}}}{l_u}\right) , \quad \bar{l}_{v_z} \leq 0 . \quad (431)$$

Therefore, if the world line of an observer is an auto-parallel trajectory, k_{\perp} is orthogonal to ξ_{\perp} , and no Coriolis acceleration $\bar{l}_{a_{\eta c}}$ exists between emitter and observer then the above expression has analogous form for description of the transversal Doppler effect as the standard (longitudinal) Doppler effect in classical mechanics in 3-dimensional Euclidean space. Here, the relation for the transversal Doppler effect is valid in every (\bar{L}_n, g) -space considered as a model of a space or a space-time under the given preconditions.

7 Hubble effect

The *Hubble effect* (Hubble shift) is the Doppler shift (Doppler effect) of signal's frequency caused by the relative motion between the emitter and the observer when the explicit form of the relative velocities and of the relative accelerations are given. The *Hubble law (law of redshift)* is defined as the linear dependence of the distances to galaxies on their red shift. In more general sense, the Hubble law is the statement that the relative velocity between an observer and a particle (from the point of view of the proper frame of reference of the observer) is proportional to the distance between the observer and the particle. Usually, the Hubble effect is defined as the change of the frequency $\bar{\omega}$ under the motion of an emitter with centrifugal (centripetal) velocity v_z relatively to an observer.

Usually, the Hubble effect is related only to the centrifugal velocity of an emitter with respect to an observer on the basis of the Hubble distance-redshift

relation [13], [14] discovered in 1929 and interpreted as a result of the expansion of the universe. The explicit form of the kinematic characteristics of the centrifugal (centripetal) and Coriolis velocities and accelerations determine uniquely the Hubble effect [12].

7.1 Explicit forms of the centrifugal (centripetal) and Coriolis velocities and accelerations

Let us now consider the explicit forms of the relative velocities and of the relative accelerations determining a Doppler shift (Doppler effect).

The vector fields generating a Doppler effect could be represented into two groups with respect to their lengths \bar{l}_ϕ :

(a) Vector fields generating a standard (longitudinal) Doppler effect

- centrifugal (centripetal) part \bar{l}_{v_z} of relative centrifugal (centripetal) velocity \bar{l}_v ,
- centrifugal (centripetal) part $\bar{l}_{v_{\eta z}}$ of relative velocity $\bar{l}_{rel v_\eta}$,
- centrifugal part (centripetal) $\bar{l}_{(a_\perp)_z}$ of acceleration \bar{l}_{a_\perp} ,
- centrifugal (centripetal) part \bar{l}_{a_z} of the relative acceleration $\bar{l}_{rel a}$,
- centrifugal (centripetal) part $\bar{l}_{a_{\eta z}}$ of the relative acceleration $\bar{l}_{rel a_\eta}$,
- centrifugal (centripetal) part $\bar{l}_{(\nabla_u a)_\perp z}$ of the change of the acceleration \bar{l}_{a_\perp} .

(b) Vector fields generating a transversal Doppler effect

- Coriolis part \bar{l}_{v_c} of relative velocity \bar{l}_v ,
- Coriolis part $\bar{l}_{v_{\eta c}}$ of a relative velocity $\bar{l}_{rel v_\eta}$,
- Coriolis part $\bar{l}_{(a_\perp)_c}$ of acceleration \bar{l}_a ,
- Coriolis part \bar{l}_{a_c} of the relative acceleration $\bar{l}_{rel a}$,
- Coriolis part $\bar{l}_{a_{\eta c}}$ of the relative acceleration $\bar{l}_{rel a_\eta}$,
- Coriolis part $\bar{l}_{(\nabla_u a)_\perp c}$ of the change $\bar{l}_{(\nabla_u a)_\perp}$ of the acceleration \bar{l}_a .

7.1.1 Explicit form of the relative velocities and accelerations generating a standard (longitudinal) Doppler effect

The relative velocities have two essential components: $rel v = v_z + v_c$, $rel v_\eta = v_{\eta z} + v_{\eta c}$.

(a) Relative centrifugal (centripetal) velocity v_z

$$v_z = \frac{g(rel v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp, \quad (432)$$

could be represented in its explicit form as

$$v_z = \left[\frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp) \right] \cdot \xi_\perp , \quad (433)$$

$$v_z = \mp l_{v_z} \cdot n_\perp = H \cdot l_{\xi_\perp} \cdot n_\perp = H \cdot \xi_\perp , \quad (434)$$

where

$$H = \frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp) , \quad (435)$$

$$l_{v_z} = \mp H \cdot l_{\xi_\perp} , \quad \bar{l}_{v_z} = \frac{dl}{l_{\xi_\perp}} \cdot l_{v_z} = \mp H \cdot dl . \quad (436)$$

The last (above) expressions for \bar{l}_{v_z} and for \bar{v}_z

$$\bar{l}_{v_z} = \mp H \cdot dl , \quad \bar{v}_z = \frac{dl}{l_{\xi_\perp}} \cdot v_z = H \cdot dl \cdot n_\perp \quad (437)$$

are the well known relations called standard (longitudinal) Hubble law: the centrifugal (centripetal) relative velocity \bar{v}_z is proportional to the distance dl between an emitter and a detector (observer). This form of the Hubble law is a very special form of the law for the case when only the centrifugal (centripetal) relative velocity is taken into account.

(b) Centrifugal (centripetal) part $\bar{l}_{v_{\eta z}}$ of a relative velocity $\bar{l}_{rel v_\eta}$

$$v_{\eta z} = \mp g_{(rel v_\eta, n_\perp)} \cdot n_\perp = \mp l_{\xi_\perp} \cdot d(n_\perp, m_\perp) \cdot n_\perp = \quad (438)$$

$$= \mp l_{v_{\eta z}} \cdot n_\perp , \quad (439)$$

$$\begin{aligned} d(n_\perp, m_\perp) &= \sigma(n_\perp, m_\perp) + \omega(n_\perp, m_\perp) + \frac{1}{n-1} \cdot \theta \cdot h_u(n_\perp, m_\perp) = \\ &= \sigma(n_\perp, m_\perp) + \omega(n_\perp, m_\perp) , \end{aligned} \quad (440)$$

$$h_u(n_\perp, m_\perp) = g(n_\perp, m_\perp) - \frac{1}{\pm l_u^2} \cdot g(u, n_\perp) \cdot g(u, m_\perp) = \quad (441)$$

$$= g(n_\perp, m_\perp) = 0 , \quad (442)$$

$$v_{\eta z} = \mp l_{v_{\eta z}} \cdot n_\perp = H_{\eta z} \cdot l_{\xi_\perp} \cdot n_\perp = \mp l_{\xi_\perp} \cdot d(n_\perp, m_\perp) \cdot n_\perp \quad (443)$$

$$\begin{aligned} H_{\eta z} &= \mp d(n_\perp, m_\perp) = \\ &= \mp [\sigma(n_\perp, m_\perp) + \omega(n_\perp, m_\perp)] = H_c , \end{aligned} \quad (444)$$

$$H_{\eta z} = H_c , \quad (445)$$

could be represented in its explicit form as

$$l_{v_{\eta z}} = \mp H_c \cdot l_{\xi_\perp} , \quad \bar{l}_{v_{\eta z}} = \frac{dl}{l_{\xi_\perp}} \cdot l_{v_{\eta z}} = \mp H_c \cdot dl , \quad (446)$$

$$\bar{l}_{v_{\eta z}} = \mp H_c \cdot dl , \quad \bar{v}_{\eta z} = \frac{dl}{l_{\xi_\perp}} \cdot v_{\eta z} = H_c \cdot dl \cdot n_\perp . \quad (447)$$

(c) Centrifugal part (centripetal) $\bar{l}_{(a_\perp)_z}$ of acceleration \bar{l}_{a_\perp}

$$a_\perp = \mp g(a_\perp, n_\perp) \cdot n_\perp + \bar{g}[h_{\xi_\perp}(a_\perp)] , \quad (448)$$

$$(a_\perp)_z = \mp l_{(a_\perp)_z} \cdot n_\perp , \quad (449)$$

$$\begin{aligned} g(a_\perp, n_\perp) &= l_{(a_\perp)_z} = g(\bar{g}[h_u(a)], n_\perp) = \\ &= g_{ij} \cdot g^{ik} \cdot h_{kl} \cdot a^l \cdot n_\perp^j = g_j^k \cdot h_{kl} \cdot a^l \cdot n_\perp^j = \\ &= h_{jl} \cdot a^l \cdot n_\perp^j = h_u(n_\perp, a) = g(n_\perp, a) , \end{aligned} \quad (450)$$

$$l_{(a_\perp)_z} = g(a_\perp, n_\perp) , \quad \bar{l}_{(a_\perp)_z} = \frac{dl}{l_{\xi_\perp}} \cdot l_{(a_\perp)_z} = \frac{dl}{l_{\xi_\perp}} \cdot g(a_\perp, n_\perp) . \quad (451)$$

(d) Centrifugal (centripetal) part \bar{l}_{a_z} of the relative acceleration $\bar{l}_{rel a}$

$$\begin{aligned} a_z &= \mp l_{\xi_\perp} \cdot A(n_\perp, n_\perp) \cdot n_\perp = \mp A(n_\perp, n_\perp) \cdot \xi_\perp = \mp l_{a_z} \cdot n_\perp = \\ &= \bar{q} \cdot \xi_\perp = \bar{q} \cdot l_{\xi_\perp} \cdot n_\perp , \end{aligned} \quad (452)$$

$$g(a_z, n_\perp) = l_{a_z} = l_{\xi_\perp} \cdot A(n_\perp, n_\perp) = \mp \bar{q} \cdot l_{\xi_\perp} , \quad (453)$$

$$A(n_\perp, n_\perp) = \mp \bar{q} , \quad (454)$$

$$\bar{q} = \frac{1}{n-1} \cdot U \mp {}_s D(n_\perp, n_\perp) , \quad (455)$$

$$a_z = \bar{q} \cdot l_{\xi_\perp} \cdot n_\perp = \mp l_{a_z} \cdot n_\perp , \quad (456)$$

$$l_{a_z} = \mp \bar{q} \cdot l_{\xi_\perp} , \quad (457)$$

$$\bar{l}_{a_z} = \mp \bar{q} \cdot \frac{dl}{l_{\xi_\perp}} \cdot l_{\xi_\perp} = \mp \bar{q} \cdot dl , \quad (458)$$

$$\bar{a}_z = \frac{dl}{l_{\xi_\perp}} \cdot a_z = \mp \bar{l}_{a_z} \cdot n_\perp = \bar{q} \cdot dl \cdot n_\perp . \quad (459)$$

The last (above) two relations for \bar{a}_z and \bar{l}_{a_z} represent the part of the Hubble effect generated by the centrifugal (centripetal) part \bar{l}_{a_z} of the acceleration $rel a$. The centrifugal (centripetal) acceleration is proportional to the distance dl between an emitter and a detector (observer).

(e) Centrifugal (centripetal) part $\bar{l}_{a_{\eta z}}$ of the relative acceleration $\bar{l}_{rel a_\eta}$

$$\begin{aligned} a_{\eta z} &= \mp g(rel a_\eta, n_\perp) \cdot n_\perp = \mp l_{\xi_\perp} \cdot A(m_\perp, n_\perp) \cdot n_\perp = \mp l_{a_{\eta z}} \cdot n_\perp = \\ &= \bar{q}_\eta \cdot l_{\xi_\perp} \cdot n_\perp = \bar{q}_\eta \cdot \xi_\perp , \end{aligned} \quad (460)$$

$$g(a_{\eta z}, n_\perp) = l_{a_{\eta z}} = l_{\xi_\perp} \cdot A(m_\perp, n_\perp) = \mp \bar{q}_\eta \cdot l_{\xi_\perp} , \quad (461)$$

$$A(n_\perp, m_\perp) = \mp \bar{q}_\eta , \quad (462)$$

$$\bar{q}_\eta = \mp A(m_\perp, n_\perp) = \mp [{}_s D(m_\perp, n_\perp) + W(m_\perp, n_\perp)] = \bar{q}_c , \quad (463)$$

$$\bar{l}_{a_{\eta z}} = \mp \bar{q}_c \cdot \frac{dl}{l_{\xi_\perp}} \cdot l_{\xi_\perp} = \mp \bar{q}_c \cdot dl , \quad (464)$$

$$\bar{a}_{\eta z} = \frac{dl}{l_{\xi_\perp}} \cdot a_{\eta z} = \mp \bar{l}_{a_{\eta z}} \cdot n_\perp = \bar{q}_c \cdot dl \cdot n_\perp . \quad (465)$$

(f) Centrifugal (centripetal) part $\bar{l}_{(\nabla_u a)_{\perp z}}$ of the change of the acceleration $\bar{l}_{a_{\perp}}$

$$(\nabla_u a)_{\perp z} = \mp g((\nabla_u a)_{\perp}, n_{\perp}) \cdot n_{\perp} = \mp l_{(\nabla_u a)_{\perp z}} \cdot n_{\perp} , \quad (466)$$

$$\begin{aligned} l_{(\nabla_u a)_{\perp z}} &= g((\nabla_u a)_{\perp}, n_{\perp}) = g(\bar{g}[h_u(\nabla_u a)], n_{\perp}) = \\ &= h_u(n_{\perp}, \nabla_u a) , \end{aligned} \quad (467)$$

$$\bar{l}_{(\nabla_u a)_{\perp z}} = \frac{dl}{l_{\xi_{\perp}}} \cdot l_{(\nabla_u a)_{\perp z}} = \frac{dl}{l_{\xi_{\perp}}} \cdot g(\nabla_u a_{\perp}, n_{\perp}) . \quad (468)$$

7.1.2 Explicit form of the relative velocities and accelerations generating a transversal Doppler effect

(a) Coriolis part \bar{l}_{v_c} of relative Coriolis velocity \bar{l}_v

$$v_c = \bar{g}[h_{\xi_{\perp}}(rel v)] \quad (469)$$

could be represented in its explicit form as

$$\begin{aligned} v_c &= \bar{g}[\sigma(\xi_{\perp})] - \frac{\sigma(\xi_{\perp}, \xi_{\perp})}{g(\xi_{\perp}, \xi_{\perp})} \cdot \xi_{\perp} = \\ &= \mp l_{v_c} \cdot m_{\perp} = \\ &= l_{\xi_{\perp}} \cdot \bar{g}[\sigma(n_{\perp})] \pm \sigma(n_{\perp}, n_{\perp}) \cdot l_{\xi_{\perp}} \cdot n_{\perp} + l_{\xi_{\perp}} \cdot \bar{g}[\omega(n_{\perp})] = \\ &= H_c \cdot l_{\xi_{\perp}} \cdot m_{\perp} . \end{aligned} \quad (470)$$

By the use of the relations

$$g(v_c, m_{\perp}) = \mp l_{v_c} \cdot g(m_{\perp}, m_{\perp}) = l_{v_c} , \quad (471)$$

$$g(v_c, m_{\perp}) = l_{\xi_{\perp}} \cdot g(\bar{g}[\sigma(n_{\perp})], m_{\perp}) + l_{\xi_{\perp}} \cdot g(\bar{g}[\omega(n_{\perp})], m_{\perp}) = l_{v_c} , \quad (472)$$

$$\begin{aligned} g(\bar{g}[\sigma(n_{\perp})], m_{\perp}) &= g_{ij}^{-1} \cdot g^{ik} \cdot \sigma_{kl} \cdot n_{\perp}^{\bar{l}} \cdot m_{\perp}^{\bar{j}} = \\ &= g_j^k \cdot \sigma_{kl} \cdot n_{\perp}^{\bar{l}} \cdot m_{\perp}^{\bar{j}} = \sigma_{jl} \cdot m_{\perp}^{\bar{j}} \cdot n_{\perp}^{\bar{l}} = \\ &= \sigma(m_{\perp}, n_{\perp}) , \end{aligned} \quad (473)$$

$$g(\bar{g}[\omega(n_{\perp})], m_{\perp}) = \omega(m_{\perp}, n_{\perp}) , \quad (474)$$

we obtain the expressions for l_{v_c} and v_c respectively

$$l_{v_c} = [\sigma(m_{\perp}, n_{\perp}) + \omega(m_{\perp}, n_{\perp})] \cdot l_{\xi_{\perp}} , \quad (475)$$

$$\begin{aligned} v_c &= \mp l_{v_c} \cdot m_{\perp} = H_c \cdot l_{\xi_{\perp}} \cdot m_{\perp} = \\ &= \mp [\sigma(m_{\perp}, n_{\perp}) + \omega(m_{\perp}, n_{\perp})] \cdot l_{\xi_{\perp}} \cdot m_{\perp} , \end{aligned} \quad (476)$$

where

$$H_c = \mp [\sigma(m_{\perp}, n_{\perp}) + \omega(m_{\perp}, n_{\perp})] . \quad (477)$$

Then

$$\bar{l}_{v_c} = \frac{dl}{l_{\xi_\perp}} \cdot l_{v_c} = \mp H_c \cdot dl , \quad (478)$$

$$\bar{l}_{v_c} = \mp H_c \cdot dl . \quad (479)$$

The last (above) expressions for \bar{l}_{v_c} and for \bar{v}_c

$$\bar{l}_{v_c} = \mp H_c \cdot dl , \quad \bar{v}_c = \frac{dl}{l_{\xi_\perp}} \cdot v_c = H_c \cdot dl \cdot m_\perp , \quad (480)$$

are the relations describing the transversal Hubble law: the Coriolis relative velocity \bar{v}_c is proportional to the distance dl between an emitter and a detector (observer). This form of the Hubble law is a very special form of the law for the case when only the Coriolis relative velocity is taken into account.

(b) Coriolis part $\bar{l}_{v_{\eta c}}$ of a relative velocity $\bar{l}_{rel v_\eta}$

$$v_{\eta c} = l_{\xi_\perp} \cdot h^{n_\perp}[d(m_\perp)] = \mp l_{v_{\eta c}} \cdot m_\perp = \overline{H}_c \cdot l_{\xi_\perp} \cdot m_\perp , \quad (481)$$

could be represented in its explicit form by the use of the relations

$$\begin{aligned} h^{n_\perp}[d(m_\perp)] &= (\bar{g} - \frac{1}{g(n_\perp, n_\perp)} \cdot n_\perp \otimes n_\perp)[d(m_\perp)] = \\ &= \bar{g}[d(m_\perp)] - \frac{1}{\mp 1} \cdot (n_\perp)[d(m_\perp)] \cdot n_\perp = \\ &= \bar{g}[d(m_\perp)] \pm (n_\perp)[d(m_\perp)] \cdot n_\perp = \\ &= \bar{g}[d(m_\perp)] \pm d(n_\perp, m_\perp) \cdot n_\perp , \end{aligned} \quad (482)$$

$$\begin{aligned} g(v_{\eta c}, m_\perp) &= l_{v_{\eta c}} = l_{\xi_\perp} \cdot g(\bar{g}[d(m_\perp)], m_\perp) = l_{\xi_\perp} \cdot g_{ij}^{ik} \cdot d_{kl} \cdot m_\perp^l \cdot m_\perp^j = \\ &= l_{\xi_\perp} \cdot g_j^k \cdot d_{kl} \cdot m_\perp^l \cdot m_\perp^j = l_{\xi_\perp} \cdot d_{jl} \cdot m_\perp^l \cdot m_\perp^j = l_{\xi_\perp} \cdot d(m_\perp, m_\perp) = \\ &= l_{\xi_\perp} \cdot \{[\sigma + \omega + \frac{1}{n-1} \cdot \theta \cdot h_u](m_\perp)\}(m_\perp) = \\ &= l_{\xi_\perp} \cdot [\sigma(m_\perp, m_\perp) \mp \frac{1}{n-1} \cdot \theta] = \end{aligned} \quad (484)$$

$$= \mp l_{\xi_\perp} \cdot [\frac{1}{n-1} \cdot \theta \mp \sigma(m_\perp, m_\perp)] , \quad (485)$$

$$l_{v_{\eta c}} = \mp [\frac{1}{n-1} \cdot \theta \mp \sigma(m_\perp, m_\perp)] \cdot l_{\xi_\perp} = \mp \overline{H}_c \cdot l_{\xi_\perp} , \quad (486)$$

$$\overline{H}_c = \frac{1}{n-1} \cdot \theta \mp \sigma(m_\perp, m_\perp) , \quad (487)$$

$$v_{\eta c} = \mp l_{v_{\eta c}} \cdot m_\perp = \overline{H}_c \cdot l_{\xi_\perp} \cdot m_\perp , \quad (488)$$

$$\bar{l}_{v_{\eta c}} = \frac{dl}{l_{\xi_\perp}} \cdot l_{v_{\eta c}} = \mp \overline{H}_c \cdot dl , \quad \bar{v}_{\eta c} = \frac{dl}{l_{\xi_\perp}} \cdot v_{\eta c} = \overline{H}_c \cdot dl \cdot m_\perp . \quad (489)$$

(c) Coriolis part $\bar{l}_{(a_\perp)_c}$ of acceleration \bar{l}_a . From the relations

$$(a_{\perp})_c = \bar{g}[h_{\xi_{\perp}}(a_{\perp})] , \quad (490)$$

$$g(\xi_{\perp}, (a_{\perp})_c) = 0 , \quad (a_{\perp})_c = \mp l_{(a_{\perp})_c} \cdot m_{\perp} , \quad (491)$$

$$\begin{aligned} l_{(a_{\perp})_c} &= g((a_{\perp})_c, m_{\perp}) = \\ &= g(\bar{g}[h_{\xi_{\perp}}(a_{\perp})], m_{\perp}) = g_{ij} \cdot g^{ik} \cdot (h_{\xi_{\perp}})_{kl} \cdot a_{\perp}^l \cdot m_{\perp}^j = \\ &= h_{\xi_{\perp}}(m_{\perp}, a_{\perp}) = h_{n_{\perp}}(m_{\perp}, a_{\perp}) = g(m_{\perp}, a_{\perp}) , \end{aligned} \quad (492)$$

$$\begin{aligned} h_{n_{\perp}}(m_{\perp}, a_{\perp}) &= [g \pm g(n_{\perp}) \otimes g(n_{\perp})](m_{\perp}, a_{\perp}) = \\ &= g(m_{\perp}, a_{\perp}) \pm g(n_{\perp}, m_{\perp}) \cdot g(n_{\perp}, a_{\perp}) = \end{aligned} \quad (493)$$

$$= g(m_{\perp}, a_{\perp}) , \quad (494)$$

it follows the form of $\bar{l}_{(a_{\perp})_c}$

$$l_{(a_{\perp})_c} = g(m_{\perp}, a_{\perp}) , \quad (a_{\perp})_c = \mp l_{(a_{\perp})_c} \cdot m_{\perp} = \mp g(m_{\perp}, a_{\perp}) \cdot m_{\perp} , \quad (495)$$

$$\bar{l}_{(a_{\perp})_c} = \frac{dl}{l_{\xi_{\perp}}} \cdot l_{(a_{\perp})_c} , \quad (\bar{a}_{\perp})_c = \frac{dl}{l_{\xi_{\perp}}} \cdot (a_{\perp})_c = \mp \bar{l}_{(a_{\perp})_c} \cdot m_{\perp} . \quad (496)$$

(d) Coriolis part \bar{l}_{a_c} of the relative acceleration $\bar{l}_{rel a}$. By means of the expressions

$$\begin{aligned} a_c &= \bar{g}[h_{\xi_{\perp}}(rel a)] = \bar{g}[h_{n_{\perp}}(rel a)] = l_{\xi_{\perp}} \cdot \bar{g}(h_{\xi_{\perp}})(\bar{g})[A(n_{\perp})] = \\ &= l_{\xi_{\perp}} \cdot h^{n_{\perp}}[A(n_{\perp})] = \mp l_{a_c} \cdot m_{\perp} = \bar{q}_c \cdot l_{\xi_{\perp}} \cdot m_{\perp} , \\ a_c &= l_{\xi_{\perp}} \cdot \{\bar{g}[{}_s D(n_{\perp})] \mp {}_s D(n_{\perp}, n_{\perp}) \cdot n_{\perp} + \bar{g}[W(n_{\perp})]\} = \end{aligned} \quad (497)$$

$$= \mp l_{a_c} \cdot m_{\perp} , \quad (498)$$

$$\begin{aligned} g(a_c, m_{\perp}) &= l_{a_c} = l_{\xi_{\perp}} \cdot \{g(\bar{g}[{}_s D(n_{\perp})], m_{\perp}) + g(\bar{g}[W(n_{\perp})], m_{\perp})\} , \\ g(\bar{g}[{}_s D(n_{\perp})], m_{\perp}) &= g_{ij} \cdot g^{ik} \cdot {}_s D_{kl} \cdot n_{\perp}^l \cdot m_{\perp}^j = \\ &= {}_s D_{jl} \cdot m_{\perp}^j \cdot n_{\perp}^l = {}_s D(m_{\perp}, n_{\perp}) , \end{aligned} \quad (499)$$

$$g(\bar{g}[W(n_{\perp})], m_{\perp}) = W(m_{\perp}, n_{\perp}) , \quad (500)$$

the explicit form of \bar{l}_{a_c} follows

$$l_{a_c} = l_{\xi_{\perp}} \cdot [{}_s D(m_{\perp}, n_{\perp}) + W(m_{\perp}, n_{\perp})] = \mp l_{\xi_{\perp}} \cdot \bar{q}_c , \quad (501)$$

$$a_c = \mp l_{a_c} \cdot m_{\perp} = \bar{q}_c \cdot l_{\xi_{\perp}} \cdot m_{\perp} , \quad (502)$$

$$\bar{q}_c = \mp [{}_s D(m_{\perp}, n_{\perp}) + W(m_{\perp}, n_{\perp})] , \quad (503)$$

$$\bar{l}_{a_c} = \frac{dl}{l_{\xi_{\perp}}} \cdot l_{a_c} = \mp \bar{q}_c \cdot dl , \quad \bar{a}_c = \frac{dl}{l_{\xi_{\perp}}} \cdot a_c = \bar{q}_c \cdot dl \cdot m_{\perp} . \quad (504)$$

(e) Coriolis part $\bar{l}_{a_{\eta c}}$ of the relative acceleration $\bar{l}_{rel a_{\eta}}$. By the use of the relations

$$\begin{aligned} a_{\eta c} &= \bar{g}[h_{\xi_{\perp}}(rel a_{\eta})] = \bar{g}[h_{n_{\perp}}(rel a_{\eta})] = l_{\xi_{\perp}} \cdot h^{n_{\perp}}[A(m_{\perp})] = \\ &= \mp l_{a_{\eta c}} \cdot m_{\perp} , \end{aligned} \quad (505)$$

$$\begin{aligned} a_{\eta c} &= l_{\xi_{\perp}} \cdot h^{n_{\perp}}[A(m_{\perp})] = \\ &= \mp l_{a_{\eta c}} \cdot m_{\perp} , \end{aligned} \quad (506)$$

$$\begin{aligned} h^{n_{\perp}}[A(m_{\perp})] &= (\bar{g} \pm n_{\perp} \otimes n_{\perp})[{}_s D(m_{\perp}) + W(m_{\perp}) + \frac{1}{n-1} \cdot U \cdot h_u(m_{\perp})] = \\ &= \bar{g}[{}_s D(m_{\perp})] + \bar{g}[W(m_{\perp})] \pm (n_{\perp})_s D(m_{\perp}) \cdot n_{\perp} \pm \\ &\quad \pm (n_{\perp})[W(m_{\perp})] \cdot n_{\perp} , \end{aligned} \quad (507)$$

$$\begin{aligned} g(a_{\eta c}, m_{\perp}) &= l_{a_{\eta c}} = \\ &= l_{\xi_{\perp}} \cdot g(\bar{g}[{}_s D(m_{\perp})], m_{\perp}) + g(\bar{g}[W(m_{\perp})], m_{\perp}) = \\ &= l_{\xi_{\perp}} \cdot [{}_s D(m_{\perp}, m_{\perp}) + W(m_{\perp}, m_{\perp})] = \\ &= l_{\xi_{\perp}} \cdot {}_s D(m_{\perp}, m_{\perp}) , \end{aligned} \quad (508)$$

$$a_{\eta c} = \mp l_{a_{\eta c}} \cdot m_{\perp} = \bar{q}_{\eta c} \cdot l_{\xi_{\perp}} \cdot m_{\perp} , \quad (509)$$

$$\bar{q}_{\eta c} = \mp {}_s D(m_{\perp}, m_{\perp}) , \quad l_{a_{\eta c}} = \mp \bar{q}_{\eta c} \cdot l_{\xi_{\perp}} , \quad (510)$$

$$\bar{l}_{a_{\eta c}} = \frac{dl}{l_{\xi_{\perp}}} \cdot l_{a_{\eta c}} = \mp \bar{q}_{\eta c} \cdot dl , \quad \bar{a}_{\eta c} = \frac{dl}{l_{\xi_{\perp}}} \cdot a_{\eta c} = \bar{q}_{\eta c} \cdot dl \cdot m_{\perp} . \quad (511)$$

(f) Coriolis part $\bar{l}_{(\nabla_u a)_{\perp c}}$ of the change $\bar{l}_{(\nabla_u a)_{\perp}}$ of the acceleration \bar{l}_a . By means of the expressions

$$(\nabla_u a)_{\perp c} = \bar{g}[h_{\xi_{\perp}}(\nabla_u a)_{\perp}] = \mp l_{(\nabla_u a)_{\perp c}} \cdot m_{\perp} , \quad (512)$$

$$g(\xi_{\perp}, (\nabla_u a)_{\perp c}) = 0 , \quad (513)$$

$$\bar{g}[h_{\xi_{\perp}}(\nabla_u a)_{\perp}] = \bar{g}[h_{n_{\perp}}(\nabla_u a)_{\perp}] = \mp l_{(\nabla_u a)_{\perp c}} \cdot m_{\perp} , \quad (514)$$

$$\begin{aligned} g((\nabla_u a)_{\perp c}, m_{\perp}) &= l_{(\nabla_u a)_{\perp c}} = g(\bar{g}[h_{n_{\perp}}(\nabla_u a)_{\perp}], m_{\perp}) = \\ &= g(\bar{g}[h_{n_{\perp}} \bar{g}[h_u(\nabla_u a)]], m_{\perp}) = \\ &= g(\bar{g}[h_{n_{\perp}} \bar{g}[h_{n_{\parallel}}(\nabla_u a)]], m_{\perp}) = \\ &= g_{\bar{i}\bar{j}} \cdot g^{ik} \cdot (h_{n_{\perp}})_{\bar{k}\bar{l}} \cdot g^{lm} \cdot (h_{n_{\parallel}})_{\bar{m}\bar{n}} \cdot a^n{}_{;r} \cdot u^r \cdot m_{\perp}^j = \\ &= (h_{n_{\perp}})_{\bar{j}\bar{l}} \cdot m_{\perp}^j \cdot g^{lm} \cdot (h_{n_{\parallel}})_{\bar{m}\bar{n}} \cdot a^n{}_{;r} \cdot u^r = \\ &= g_{\bar{j}\bar{l}} \cdot g^{lm} \cdot m_{\perp}^j \cdot (h_{n_{\parallel}})_{\bar{m}\bar{n}} \cdot a^n{}_{;r} \cdot u^r = \\ &= (h_{n_{\parallel}})_{\bar{j}\bar{n}} \cdot m_{\perp}^j \cdot a^n{}_{;r} \cdot u^r = \\ &= h_{n_{\parallel}}(m_{\perp}, \nabla_u a) = h_u(m_{\perp}, \nabla_u a) = g(m_{\perp}, \nabla_u a) , \end{aligned} \quad (515)$$

$$l_{(\nabla_u a)_{\perp c}} = h_{n_{\perp}}(m_{\perp}, \nabla_u a) = g(m_{\perp}, \nabla_u a) , \quad (516)$$

$$\begin{aligned} (\nabla_u a)_{\perp c} &= \mp l_{(\nabla_u a)_{\perp c}} \cdot m_{\perp} = \mp h_{n_{\perp}}(m_{\perp}, \nabla_u a) \cdot m_{\perp} = \\ &= \mp g(m_{\perp}, \nabla_u a) , \end{aligned} \quad (517)$$

$$\bar{l}_{(\nabla_u a)_{\perp c}} = \frac{dl}{l_{\xi_{\perp}}} \cdot l_{(\nabla_u a)_{\perp c}} , \quad (\nabla_u \bar{a})_{\perp} = \frac{dl}{l_{\xi_{\perp}}} \cdot (\nabla_u a)_{\perp c} . \quad (518)$$

7.2 Standard (longitudinal) Hubble effect (Hubble shift)

The standard (longitudinal) Hubble effect (Hubble shift) corresponds to the standard (longitudinal) Doppler effect (Doppler shift). Only the different types of velocities and accelerations generating the standard Doppler effect are given in their explicit form by means of the corresponding Hubble functions and acceleration parameters.

If the vector field k_\perp is collinear to the vector field ξ_\perp determining the proper frame of reference, i.e. if

$$k_\perp = \mp l_{k_\perp} \cdot n_\perp \quad , \quad \cos \theta = \pm 1 \quad , \quad \sin \theta = 0 \quad , \quad (519)$$

the frequency of the emitter $\bar{\omega}$ and the frequency ω detected by the observer are related to each other by the expression

$$\bar{\omega} = \omega \cdot \left[1 - \frac{\bar{l}_{v_z}}{l_u} - \frac{l_{\xi_\perp}}{l_u^2} \cdot \bar{l}_{(a_\perp)_z} + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot (\bar{l}_{a_z} + \frac{l_{\xi_\perp}}{l_u} \cdot \bar{l}_{(\nabla_u a)_\perp z}) \right] \quad . \quad (520)$$

If we now replace the velocity \bar{l}_{v_z} and the accelerations $\bar{l}_{(a_\perp)_z}$, \bar{l}_{a_z} , and $\bar{l}_{(\nabla_u a)_\perp z}$ with their corresponding explicit forms

$$\bar{l}_{v_z} = \mp H \cdot dl \quad , \quad (521)$$

$$\bar{l}_{a_z} = \mp \bar{q} \cdot dl \quad , \quad (522)$$

$$\bar{l}_{(a_\perp)_z} = \frac{dl}{l_{\xi_\perp}} \cdot g(a_\perp, n_\perp) \quad , \quad (523)$$

$$\bar{l}_{(\nabla_u a)_\perp z} = \frac{dl}{l_{\xi_\perp}} \cdot g(\nabla_u a_\perp, n_\perp) \quad , \quad (524)$$

then we obtain the relation between $\bar{\omega}$ and ω in the form

$$\begin{aligned} \bar{\omega} &= \omega \cdot \left[1 - \frac{\bar{l}_{v_z}}{l_u} - \frac{l_{\xi_\perp}}{l_u^2} \cdot \bar{l}_{(a_\perp)_z} + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot (\bar{l}_{a_z} + \frac{l_{\xi_\perp}}{l_u} \cdot \bar{l}_{(\nabla_u a)_\perp z}) \right] = \\ &= \omega \cdot \left\{ 1 - \frac{1}{l_u} \cdot [\bar{l}_{v_z} + \frac{l_{\xi_\perp}}{l_u} \cdot \bar{l}_{(a_\perp)_z} - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot (\bar{l}_{a_z} + \frac{l_{\xi_\perp}}{l_u} \cdot \bar{l}_{(\nabla_u a)_\perp z})] \right\} = \\ &= \omega \cdot \left\{ 1 - \frac{1}{l_u} \cdot [\mp H \cdot dl + \frac{l_{\xi_\perp}}{l_u} \cdot \frac{dl}{l_{\xi_\perp}} \cdot g(a_\perp, n_\perp) - \right. \\ &\quad \left. - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot (\mp \bar{q} \cdot dl + \frac{l_{\xi_\perp}}{l_u} \cdot \frac{dl}{l_{\xi_\perp}} \cdot g(\nabla_u a_\perp, n_\perp))] \right\} \quad , \quad (525) \end{aligned}$$

$$\begin{aligned} \bar{\omega} &= \omega \cdot \left\{ 1 - \frac{1}{l_u} \cdot [\mp H \cdot dl + \frac{dl}{l_u} \cdot g(a_\perp, n_\perp) - \right. \\ &\quad \left. - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot (\mp \bar{q} \cdot dl + \frac{dl}{l_u} \cdot g(\nabla_u a_\perp, n_\perp))] \right\} \quad , \quad (526) \end{aligned}$$

$$\begin{aligned} \bar{\omega} &= \omega \cdot \left\{ 1 + \frac{1}{l_u} \cdot [\pm H \cdot dl - \frac{dl}{l_u} \cdot g(a_\perp, n_\perp) - \right. \\ &\quad \left. - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot (\pm \bar{q} \cdot dl - \frac{dl}{l_u} \cdot g(\nabla_u a_\perp, n_\perp))] \right\} \quad (527) \end{aligned}$$

$$\bar{\omega} = \omega \cdot \left\{ 1 \pm \frac{1}{l_u} \cdot [H \cdot dl \mp \frac{dl}{l_u} \cdot g(a_\perp, n_\perp) - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot (\bar{q} \cdot dl \mp \frac{dl}{l_u} \cdot g(\nabla_u a_\perp, n_\perp))] \right\} . \quad (528)$$

Special case: Auto-parallel motion of the observer: $\nabla_u u = a = 0$: $\bar{l}_{(a_\perp)_z} = 0$, $\bar{l}_{(\nabla_u a)_\perp z} = 0$,

$$\begin{aligned} \bar{\omega} &= \omega \cdot \left[1 - \frac{\bar{l}_{v_z}}{l_u} + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot \bar{l}_{a_z} \right] , \\ \bar{\omega} &= \omega \cdot \left[1 \pm \frac{1}{l_u} \cdot (H \cdot dl - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot \bar{q} \cdot dl) \right] = \\ &= \omega \cdot \left[1 \pm \frac{1}{l_u} \cdot (H - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot \bar{q}) \cdot dl \right] . \end{aligned} \quad (529)$$

If the world line of an observer is an auto-parallel trajectory and k_\perp is collinear to ξ_\perp then the change of the frequency of the emitter $\bar{\omega}$ depends on the Hubble function H and the acceleration parameter \bar{q} .

Special case: $\nabla_u u = a = 0$, $k_\perp = \mp l_{k_\perp} \cdot n_\perp$, $\bar{l}_{a_z} = 0$:

$$\bar{\omega} = \omega \cdot \left(1 - \frac{\bar{l}_{v_z}}{l_u} \right) , \quad \bar{l}_{v_z} \leq 0 .$$

$$\bar{\omega} = \omega \cdot \left(1 \pm \frac{1}{l_u} \cdot H \cdot dl \right) . \quad (530)$$

Therefore, if the world line of an observer is an auto-parallel trajectory, k_\perp is collinear to ξ_\perp , and no centrifugal (centripetal) acceleration \bar{l}_{a_z} exists between emitter and observer then the above expression has the well known form for description of the standard Hubble effect in relativistic astrophysics. Here, this relation is valid in every (\bar{L}_n, g) -space considered as a model of a space or a space-time under the given preconditions.

7.2.1 Standard (longitudinal) Hubble shift frequency parameter z

The relative difference between both the frequencies (emitted $\bar{\omega}$ and detected ω)

$$\frac{\bar{\omega} - \omega}{\omega} := z = \pm \bar{C} , \quad (531)$$

under the condition $k_\perp = \mp l_{k_\perp} \cdot n_\perp$ appears in the form

$$\begin{aligned} \frac{\bar{\omega} - \omega}{\omega} : &= z = \pm \bar{C} = \pm \frac{1}{l_u} \cdot [H \cdot dl \mp \frac{dl}{l_u} \cdot g(a_\perp, n_\perp) - \\ &- \frac{1}{2} \cdot \frac{dl}{l_u} \cdot (\bar{q} \cdot dl \mp \frac{dl}{l_u} \cdot g(\nabla_u a_\perp, n_\perp))] , \end{aligned} \quad (532)$$

where

$$\bar{\omega} = (1 + z) \cdot \omega , \quad (533)$$

$$z = \pm \frac{1}{l_u} \cdot [H \cdot dl \mp \frac{dl}{l_u} \cdot g(a_\perp, n_\perp) - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot (\bar{q} \cdot dl \mp \frac{dl}{l_u} \cdot g(\nabla_u a_\perp, n_\perp))] . \quad (534)$$

The quantity z could be denoted as *observed standard (longitudinal) Hubble shift frequency parameter*. If $z = 0$ then there will be no difference between the emitted and the detected frequencies, i. e. $\bar{\omega} = \omega$. This will be the case when the emitter and observer (detector) are at rest to each other, i.e. when no relative velocities and relative accelerations occur, when centrifugal (centripetal) relative velocities and accelerations do not exist, or when the centripetal (centrifugal) velocities and accelerations compensate each other under the condition

$$H \mp \frac{1}{l_u} \cdot g(a_\perp, n_\perp) - \frac{1}{2} \cdot \frac{1}{l_u} \cdot (\bar{q} \mp \frac{1}{l_u} \cdot g(\nabla_u a_\perp, n_\perp)) \cdot dl = 0 \quad . \quad (535)$$

If $z > 0$ the observed Hubble shift frequency parameter is called *longitudinal Hubble red shift*. If $z < 0$ the observed Hubble shift frequency parameter is called *longitudinal Hubble blue shift*. If $\bar{\omega}$ and ω are known the observed Hubble shift frequency parameter z could be found. If ω and z are given then the corresponding $\bar{\omega}$ could be estimated.

On the other side, from the explicit form of z

$$z = \pm \frac{1}{l_u} \cdot [H \cdot dl \mp \frac{dl}{l_u} \cdot g(a_\perp, n_\perp) - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot (\bar{q} \cdot dl \mp \frac{dl}{l_u} \cdot g(\nabla_u a_\perp, n_\perp))] \quad (536)$$

if we consider the explicit form of the Hubble function H and of the acceleration parameter \bar{q} we could find the relation between the observed shift frequency parameter z and the kinematic characteristics of the relative velocity such as expansion and shear velocities and accelerations.

Special case: Auto-parallel motion of the observer: $\nabla_u u = a = 0$: $\bar{l}_{(a_\perp)_z} = 0$, $\bar{l}_{(\nabla_u a)_\perp z} = 0$,

$$\bar{\omega} = \omega \cdot [1 - \frac{\bar{l}_{v_z}}{l_u} + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot \bar{l}_{a_z}] \quad ,$$

$$z = \pm \frac{1}{l_u} \cdot (H \cdot dl - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot \bar{q} \cdot dl) = \pm \frac{1}{l_u} \cdot (H - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot \bar{q}) \cdot dl \quad . \quad (537)$$

If the world line of an observer is an auto-parallel trajectory and k_\perp is collinear to ξ_\perp then the observed Hubble shift frequency parameter z depends on the Hubble function H and the acceleration parameter \bar{q} as well as on the absolute value l_u of the velocity of the signal and on the space distance dl propagated by the signal.

Special case: $\nabla_u u = a = 0$, $k_\perp = \mp l_{k_\perp} \cdot n_\perp$, $\bar{l}_{a_z} = 0$:

$$\bar{\omega} = \omega \cdot (1 - \frac{\bar{l}_{v_z}}{l_u}) \quad , \quad \bar{l}_{v_z} \leq 0 \quad .$$

$$z = \pm \frac{1}{l_u} \cdot H \cdot dl \quad . \quad (538)$$

Remark. In relativistic physics ($l_u = c$, 1) the last (above) relation is also called Hubble law.

If we express in this special case the observed longitudinal Hubble shift frequency parameter z in its infinitesimal form

$$z = \frac{\bar{\omega} - \omega}{\omega} = \frac{d\omega}{\omega} = \pm \frac{1}{l_u} \cdot H \cdot dl \quad (539)$$

then we can find the change of the frequency ω for a global distance l propagated by a signal for finite proper time interval of an observer

$$\begin{aligned} \int \frac{d\omega}{\omega} &= \pm \int \frac{1}{l_u} \cdot H \cdot dl \quad , \\ \log \omega &= \pm \int \frac{1}{l_u} \cdot H \cdot dl + \text{const.} \quad , \\ \omega &= \omega_0 \cdot \exp(\pm \int \frac{1}{l_u} \cdot H \cdot dl) \quad , \quad \omega_0 = \text{const.} \end{aligned} \quad (540)$$

In the relativistic astrophysics, it is assumed that $l_u = \text{const.} = c$ or 1. The Hubble function is also assumed to be a constant function $H = H_0 = \text{const.}$

Then

$$\bar{\omega} = \omega_0 \cdot \exp(\pm \frac{1}{l_u} \cdot H_0 \cdot l) = \omega_0 \cdot \exp(\pm H_0 \cdot \frac{l}{l_u}) = \omega_0 \cdot \exp(\pm H_0 \cdot \tau) \quad , \quad (541)$$

where l_u is the absolute value of the velocity of a signal, τ is the proper time interval of the observer for which a signal propagates from the emitter to the observer, and l is the space distance covered by the signal from the emitter to the observer.

Therefore, if the world line of an observer is an auto-parallel trajectory, k_\perp is collinear to ξ_\perp , and no centrifugal (centripetal) acceleration \bar{l}_{a_z} exists between an emitter and an observer then the above expression has the well known form for description of the standard Hubble effect in relativistic astrophysics. Here, the relation is valid in every (\bar{L}_n, g) -space considered as a model of a space or a space-time under the given preconditions.

7.3 Transversal Hubble effect (Hubble shift)

The transversal Hubble effect (Hubble shift) corresponds to the transversal Doppler effect (Doppler shift). Only the different types of velocities and accelerations generating the transversal Doppler effect are given in their explicit form by means of the corresponding Hubble functions and acceleration parameters.

If the vector field k_\perp is collinear to the vector field ξ_\perp determining the proper frame of reference, i.e. if

$$k_\perp = \mp l_{k_\perp} \cdot n_\perp \quad , \quad \cos \theta = \pm 1 \quad , \quad \sin \theta = 0 \quad , \quad (542)$$

the frequency of the emitter $\bar{\omega}$ and the frequency ω detected by the observer are related to each other by the expression

$$\begin{aligned} \bar{\omega} &= \omega \cdot [1 - \frac{1}{l_u} \cdot \bar{l}_{v_{\eta c}} - \frac{l_{\xi_\perp}}{l_u^2} \cdot \bar{l}_{(a_\perp)_c} + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot (\bar{l}_{a_{\eta c}} + \frac{l_{\xi_\perp}}{l_u} \cdot l_{(\nabla_u a)_\perp c})] = \\ &= \omega \cdot \{1 - \frac{1}{l_u} \cdot [\bar{l}_{v_{\eta c}} + \frac{l_{\xi_\perp}}{l_u} \cdot \bar{l}_{(a_\perp)_c} - \\ &\quad - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot (\bar{l}_{a_{\eta c}} + \frac{l_{\xi_\perp}}{l_u} \cdot l_{(\nabla_u a)_\perp c})]\} \quad . \end{aligned} \quad (543)$$

If we now replace the velocity $\bar{l}_{v_{\eta c}}$ and the accelerations $\bar{l}_{(a_{\perp})_c}$, $\bar{l}_{a_{\eta c}}$, and $\bar{l}_{(\nabla_u a)_{\perp c}}$ with their corresponding explicit forms

$$\bar{l}_{v_{\eta c}} = \mp \bar{H}_c \cdot dl \quad , \quad (544)$$

$$\bar{l}_{(a_{\perp})_c} = \frac{dl}{l_{\xi_{\perp}}} \cdot l_{(a_{\perp})_c} = \frac{dl}{l_{\xi_{\perp}}} \cdot g(m_{\perp}, a_{\perp}) \quad , \quad (545)$$

$$\bar{l}_{a_{\eta c}} = \mp \bar{q}_{\eta c} \cdot dl \quad , \quad (546)$$

$$\bar{l}_{(\nabla_u a)_{\perp c}} = \frac{dl}{l_{\xi_{\perp}}} \cdot l_{(\nabla_u a)_{\perp c}} = \frac{dl}{l_{\xi_{\perp}}} \cdot g(m_{\perp}, \nabla_u a) \quad , \quad (547)$$

then we obtain the relation between $\bar{\omega}$ and ω in the form

$$\begin{aligned} \bar{\omega} &= \omega \cdot \left\{ 1 - \frac{1}{l_u} \cdot [\bar{l}_{v_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \cdot \bar{l}_{(a_{\perp})_c} - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot (\bar{l}_{a_{\eta c}} + \frac{l_{\xi_{\perp}}}{l_u} \cdot l_{(\nabla_u a)_{\perp c}})] \right\} = \\ &= \omega \cdot \left\{ 1 - \frac{1}{l_u} \cdot [\mp \bar{H}_c \cdot dl + \frac{l_{\xi_{\perp}}}{l_u} \cdot \frac{dl}{l_{\xi_{\perp}}} \cdot g(m_{\perp}, a_{\perp})] - \right. \\ &\quad \left. - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot [\mp \bar{q}_{\eta c} \cdot dl + \frac{l_{\xi_{\perp}}}{l_u} \cdot \frac{dl}{l_{\xi_{\perp}}} \cdot g(m_{\perp}, \nabla_u a)] \right\} \quad , \end{aligned} \quad (548)$$

$$\bar{\omega} = \omega \cdot \left\{ 1 \pm \frac{1}{l_u} \cdot [\bar{H}_c \cdot dl \mp \frac{dl}{l_u} \cdot g(m_{\perp}, a_{\perp}) - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot (\bar{q}_{\eta c} \cdot dl \mp \frac{dl}{l_u} \cdot g(m_{\perp}, \nabla_u a))] \right\} \quad , \quad (549)$$

compared with the case of the standard (longitudinal) Hubble effect

$$\bar{\omega} = \omega \cdot \left\{ 1 \pm \frac{1}{l_u} \cdot [H \cdot dl \mp \frac{dl}{l_u} \cdot g(a_{\perp}, n_{\perp}) - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot (\bar{q} \cdot dl \mp \frac{dl}{l_u} \cdot g(\nabla_u a_{\perp}, n_{\perp}))] \right\} \quad .$$

Special case: Auto-parallel motion of the observer: $\nabla_u u = a = 0$: $\bar{l}_{(a_{\perp})_c} = 0$, $\bar{l}_{(\nabla_u a)_{\perp c}} = 0$,

$$\bar{\omega} = \omega \cdot \left[1 - \frac{1}{l_u} \cdot \bar{l}_{v_{\eta c}} + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot \bar{l}_{a_{\eta c}} \right] \quad ,$$

$$\begin{aligned} \bar{\omega} &= \omega \cdot \left[1 \pm \frac{1}{l_u} \cdot (\bar{H}_c \cdot dl - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot \bar{q}_{\eta c} \cdot dl) \right] = \\ &= \omega \cdot \left[1 \pm \frac{1}{l_u} \cdot (\bar{H}_c - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot \bar{q}_{\eta c}) \cdot dl \right] \quad . \end{aligned} \quad (550)$$

If the world line of an observer is an auto-parallel trajectory and k_{\perp} is orthogonal to ξ_{\perp} then the change of the frequency of the emitter $\bar{\omega}$ depends on the Coriolis velocity $\bar{l}_{v_{\eta c}}$ and the Coriolis acceleration $\bar{l}_{a_{\eta c}}$.

Special case: $\nabla_u u = a = 0$, $k_{\perp} = \mp l_{k_{\perp}} \cdot m_{\perp}$, $\bar{l}_{a_{\eta c}} = 0$:

$$\begin{aligned} \bar{\omega} &= \omega \cdot \left(1 - \frac{\bar{l}_{v_{\eta c}}}{l_u} \right) \quad , \quad \bar{l}_{v_{\eta z}} \leq 0 \quad , \\ \bar{\omega} &= \omega \cdot \left(1 \pm \frac{1}{l_u} \cdot \bar{H}_c \cdot dl \right) \quad . \end{aligned} \quad (551)$$

Therefore, if the world line of an observer is an auto-parallel trajectory, k_\perp is orthogonal to ξ_\perp , and no Coriolis acceleration $\bar{l}_{a_{\eta c}}$ exists between emitter and observer then the above expression has analogous form for description of the transversal Hubble effect as the standard (longitudinal) Hubble effect in relativistic astrophysics. Here, this relation for the transversal Hubble effect is valid in every (\bar{L}_n, g) -space considered as a model of a space or a space-time under the given preconditions.

7.3.1 Transversal Hubble shift frequency parameter z_c

The relative difference between both the frequencies (emitted $\bar{\omega}$ and detected ω) when a transversal Doppler effect and a transversal Hubble effect correspondingly occur could be written as

$$\frac{\bar{\omega} - \omega}{\omega} := z_c = \pm \bar{S} \quad (552)$$

and under the condition $k_\perp = \mp l_{k_\perp} \cdot n_\perp$ appears in the form

$$\begin{aligned} \frac{\bar{\omega} - \omega}{\omega} : = z_c = & \pm \frac{1}{l_u} \cdot [\bar{H}_c \cdot dl \mp \frac{dl}{l_u} \cdot g(m_\perp, a_\perp) - \\ & - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot \bar{q}_{\eta c} \cdot dl \mp \frac{dl}{l_u} \cdot g(m_\perp, \nabla_u a)] \end{aligned} \quad (553)$$

where

$$\bar{\omega} = (1 + z_c) \cdot \omega \quad , \quad (554)$$

$$\begin{aligned} z_c = & \pm \frac{1}{l_u} \cdot [\bar{H}_c \cdot dl \mp \frac{dl}{l_u} \cdot g(m_\perp, a_\perp) - \\ & - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot (\bar{q}_{\eta c} \cdot dl \mp \frac{dl}{l_u} \cdot g(m_\perp, \nabla_u a))] \quad . \end{aligned} \quad (555)$$

The quantity z_c could be denoted as *observed transversal Hubble shift frequency parameter*. If $z_c = 0$ then there will be no difference between the emitted and the detected frequencies, i. e. $\bar{\omega} = \omega$. This will be the case when the emitter and observer (detector) are at rest to each other, i.e. when no relative velocities and relative accelerations occur, when Coriolis relative velocities and Coriolis relative accelerations do not exist, or when the centripetal (centrifugal) velocities and accelerations compensate each other under the condition

$$\bar{H}_c \mp \frac{1}{l_u} \cdot g(m_\perp, a_\perp) - \frac{1}{2} \cdot \frac{1}{l_u} \cdot (\bar{q}_{\eta c} \mp \frac{1}{l_u} \cdot g(m_\perp, \nabla_u a)) \cdot dl = 0 \quad . \quad (556)$$

If $z_c > 0$ the observed transversal Hubble shift frequency parameter is called *transversal Hubble red shift*. If $z_c < 0$ the observed transversal Hubble shift frequency parameter is called *transversal Hubble's blue shift*. If $\bar{\omega}$ and ω are known the observed transversal Hubble shift frequency parameter z_c could be found. If ω and z are given then the corresponding $\bar{\omega}$ could be estimated.

On the other side, from the explicit form of z_c

$$\begin{aligned} z_c = & \pm \frac{1}{l_u} \cdot [\bar{H}_c \cdot dl \mp \frac{dl}{l_u} \cdot g(m_\perp, a_\perp) - \\ & - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot (\bar{q}_{\eta c} \cdot dl \mp \frac{dl}{l_u} \cdot g(m_\perp, \nabla_u a))] \end{aligned} \quad (557)$$

if we consider the explicit form of the Hubble function \overline{H}_c and of the acceleration parameter $\overline{q}_{\eta c}$ we could find the relation between the observed shift frequency parameter z_c and the kinematic characteristics of the relative velocity such as expansion and shear velocities and accelerations.

Special case: Auto-parallel motion of the observer: $\nabla_u u = a = 0$: $\bar{l}_{(a\perp)_z} = 0$, $\bar{l}_{(\nabla_u a)\perp z} = 0$,

$$\begin{aligned}\overline{\omega} &= \omega \cdot \left[1 - \frac{1}{l_u} \cdot \bar{l}_{v_{\eta c}} + \frac{1}{2} \cdot \frac{dl}{l_u^2} \cdot \bar{l}_{a_{\eta c}} \right], \\ z_c &= \pm \frac{1}{l_u} \cdot (\overline{H}_c - \frac{1}{2} \cdot \frac{dl}{l_u} \cdot \overline{q}_{\eta c}) \cdot dl \quad .\end{aligned}\tag{558}$$

If the world line of an observer is an auto-parallel trajectory and k_\perp is collinear to ξ_\perp then the observed transversal Hubble shift frequency parameter z_c depends on the Hubble function \overline{H}_c and the acceleration parameter $\overline{q}_{\eta c}$ as well as on the absolute value l_u of the velocity of the signal and on the space distance dl propagated by the signal.

Special case: $\nabla_u u = a = 0$, $k_\perp = \mp l_{k_\perp} \cdot n_\perp$, $\bar{l}_{a_z} = 0$:

$$\begin{aligned}\overline{\omega} &= \omega \cdot \left(1 - \frac{\bar{l}_{v_{\eta c}}}{l_u} \right) \quad , \quad \bar{l}_{v_{\eta z}} \begin{matrix} \leq \\ \geq \end{matrix} 0 \quad , \\ z_c &= \pm \frac{1}{l_u} \cdot \overline{H}_c \cdot dl \quad .\end{aligned}\tag{559}$$

If we express in this special case the observed transversal Hubble shift frequency parameter z_c in its infinitesimal form

$$z_c = \frac{\overline{\omega} - \omega}{\omega} = \frac{d\omega}{\omega} = \pm \frac{1}{l_u} \cdot \overline{H}_c \cdot dl\tag{560}$$

then we can find the change of the frequency ω for a global distance l propagated by a signal for finite proper time interval of an observer

$$\begin{aligned}\int \frac{d\omega}{\omega} &= \pm \int \frac{1}{l_u} \cdot \overline{H}_c \cdot dl \quad , \\ \log \omega &= \pm \int \frac{1}{l_u} \cdot \overline{H}_c \cdot dl + const. \quad , \\ \omega &= \omega_0 \cdot \exp\left(\pm \int \frac{1}{l_u} \cdot \overline{H}_c \cdot dl\right) \quad , \quad \omega_0 = const.\end{aligned}\tag{561}$$

In the relativistic astrophysics, it is assumed that $l_u = \text{const.} = c$ or 1. If the Hubble function is also assumed to be a constant function $\overline{H}_c = \overline{H}_{c0} = \text{const.}$

Then

$$\overline{\omega} = \omega_0 \cdot \exp\left(\pm \frac{1}{l_u} \cdot \overline{H}_{c0} \cdot l\right) = \omega_0 \cdot \exp\left(\pm \overline{H}_{c0} \cdot \frac{l}{l_u}\right) = \omega_0 \cdot \exp(\pm \overline{H}_{c0} \cdot \tau) \quad ,\tag{562}$$

where l_u is the absolute value of the velocity of a signal, τ is the proper time interval of the observer for which a signal propagates from the emitter to the

observer, and l is the space distance covered by the signal from the emitter to the observer. If $l = \text{const.}$ then the $\bar{\omega}$ will change by a constant quantity

$$\bar{\omega} = K_0 \cdot \omega \quad , \quad K_0 = \exp(\pm \bar{H}_{c0} \cdot \frac{l}{l_u}) = \exp(\pm \bar{H}_{c0} \cdot \tau) = \text{const.} \quad (563)$$

and we will observe a constant shift of the emitted frequency with respect to the observer (detector) during a time interval. If we further write K_0 in the form

$$K_0 = 1 \pm \bar{K}_0 \quad , \quad \bar{K}_0 = \text{const.},$$

the change of the frequency $\bar{\omega}$ could be represented in the forms

$$\bar{\omega} = (1 \pm \bar{K}_0) \cdot \omega \quad , \quad (564)$$

$$z_c = \pm \bar{K}_0 = \text{const.} \quad (565)$$

If the space distance between an emitter and an observer does not change but there is a Coriolis velocity $\bar{l}_{v_{\eta c}}$ between them then there will be a constant difference $z_c = \pm \bar{K}_0 = \text{const.}$ between the emitted frequency $\bar{\omega}$ and the detected frequency ω . This could be the case when an emitter sends signals to a detector and rotates along the detector at a constant space distance from it. At the same time the detector moves at an auto-parallel world line.

Therefore, if the world line of an observer is an auto-parallel trajectory, k_{\perp} is collinear to ξ_{\perp} , and no Coriolis acceleration $\bar{l}_{a_{\eta c}}$ exists between emitter and observer then the above expression could be used for description of the transversal Hubble effect in relativistic astrophysics. Here, the relation is valid in every (\bar{L}_n, g) -space considered as a model of a space or of a space-time under the given preconditions.

8 Conclusion

In the present paper we have considered the notion of null (isotropic) vector field in spaces with affine connections and metrics for describing effects caused by the relative motion between emitters and detectors in spaces with affine connections and metrics used as models of space or of space-time. On the basis of the notions of centrifugal (centripetal) and Coriolis velocities and accelerations the notions of aberration, standard (longitudinal) and transversal Doppler effects, and standard and transversal Hubble effect are introduced and considered. It is shown that the reasons for aberration, Doppler effect, and Hubble effect could be not only relative velocities between an emitter and a detector but also relative accelerations between them. It is shown that the Hubble effect is nothing more than the Doppler effect with explicitly given structures of the relative velocities and relative accelerations. By the use of the Hubble law, leading to the introduction of the Hubble effect, some connections between the kinematic characteristics of the relative velocity and the relative acceleration, on the one side, and the Doppler effects, the Hubble effect, and the aberration, on the other side, are investigated.

The aberration, the Doppler effects, and the Hubble effects are considered on the grounds of purely kinematic considerations. It should be stressed that

the Hubble functions H and \overline{H}_c are introduced on a purely kinematic basis related to the notions of relative centrifugal (centripetal) velocity and to the notions of Coriolis velocities respectively. Their dynamic interpretations in a theory of gravitation depend on the structures of the theory and the relations between the field equations and on both the functions. In this paper it is shown that notions the specialists use to apply in theories of gravitation and cosmological models could have a good kinematic grounds independent of any concrete classical field theory. Aberration, Doppler effects, and Hubble effects could be used in mechanics of continuous media and in other classical field theories in the same way as the standard Doppler effect is used in classical and relativistic mechanics.

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